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TIGHT CONTACT STRUCTURES ON FIBERED HYPERBOLIC 3-MANIFOLDS

KO HONDA, WILLIAM H. KAZEZ & GORDANA MATIĆ

Abstract

As a first step towards understanding the relationship between foliations and tight contact structures on hyperbolic 3-manifolds, we classify "extremal" tight contact structures on a surface bundle M over the circle with pseudo-Anosov monodromy. More specifically, there is exactly one tight contact structure (up to isotopy) whose Euler class, when evaluated on the fiber, equals the Euler characteristic of the fiber. This rigidity theorem is a consequence of properties of the action of pseudo-Anosov maps on the complex of curves of the fiber and a remarkable flexibility property of convex surfaces in M. Indeed, this flexibility can already be seen in surface bundles over the interval, where an analogous classification theorem is also established.

1. Introduction

The recent work of Colin [4] and the authors [18], together with the work (in progress) of Colin, Giroux, and the first author [5], suggest a very general, but rough, classification principle for tight contact structures.

Principle. If M is a closed, oriented, irreducible 3-manifold, then M carries finitely many isotopy classes of tight contact structures if and only if M is atoroidal.

The work [5] on the finiteness of tight contact structures does not give an accurate bound on the number of tight contact structures carried by an atoroidal manifold. (For example, it does not address the

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existence question.) Indeed, it is not clear, *a priori*, just what are the properties of a 3-manifold which determine the number and nature of the tight contact structures it carries.

As a first step towards understanding the classification of tight contact structures for atoroidal manifolds with infinite fundamental group, we consider hyperbolic 3-manifolds which fiber over the circle. These manifolds have enough structure to be accessible through a variety of techniques, and the relationships among the various structures they support are often extremely good predictors of similar relationships in more general 3-manifolds. The uniqueness theorem (Theorem 1.2) suggests that contact topology may ultimately be a discrete version of foliation theory. Eliashberg and Thurston [7] showed that any taut foliation may be perturbed into a universally tight contact structure. There are many foliations of a fibered hyperbolic manifold that contain a fixed fiber as a leaf, yet perturbing any of them always produces the same universally tight contact structure by Theorem 1.2.

Contact structures on fibered hyperbolic 3-manifolds are studied by splitting the manifold along a fiber and later recreating the original manifold by gluing back using the pseudo-Anosov monodromy map. This leads to an analysis of tight contact structures on the product $\Sigma\times[0,1]$, where Σ is a closed, oriented surface of genus $g>1$. There is a surprisingly small number of (isotopy classes of) tight contact structures on $\Sigma\times I$ with boundary conditions given in Theorem 1.1, and the theory for $g>1$ is quite different from the $g=1$ case treated in [11, 14]. Much of the classification on $\Sigma\times I$, at least in the cases we are led to study, can be encoded in the curve complex of Σ. Therefore, Theorem 1.2 exploits the interplay between the pseudo-Anosov monodromy and the curve complex of Σ. The fact that there are only a few distinct isotopy classes of tight contact structures on $\Sigma\times I$ (of the kind we are interested in) is in large part due to a remarkable flexibility property (Proposition 6.2) enjoyed by surfaces in $\Sigma\times I$ which are isotopic to $\Sigma\times\{t\}$. To paraphrase the result, by isotoping such a surface, we are free to choose any pair of parallel, nonseparating curves as its dividing set, provided the tight contact structure on $\Sigma\times I$ is not the I-invariant one.

Let Σ be a closed, oriented surface of genus $g>1$ and $f:\Sigma\to\Sigma$ be a pseudo-Anosov diffeomorphism. We study tight contact structures ξ on the 3-manifold $M=\Sigma\times I=\Sigma\times[0,1]$ or $M=\Sigma\times[0,1]/\sim$, given by the equivalence relation $(x,0)\sim(f(x),1)$, which satisfy the following condition:

Extremal condition. $\langle e(\xi), \Sigma_t \rangle = \pm(2g-2)$, $t \in [0,1]$, where the left-hand side refers to the Euler class of ξ evaluated on Σ_t.

Here we write $\Sigma_t = \Sigma \times \{t\}$. Tight contact structures on M satisfying this condition are said to be *extremal* for the following reason: if ξ is a tight contact structure on M, then the Bennequin inequality [1, 6] states that:

$$-(2g-2) \le \langle e(\xi), \Sigma_t \rangle \le 2g-2.$$

One of the main results of this paper is the following classification:

Theorem 1.1. *Let Σ be a closed oriented surface of genus $\geq$ 2 and $M = \Sigma \times I$. Fix dividing sets $\Gamma_{\Sigma_i} = 2\gamma_i$ (2 parallel disjoint copies of γ_i), $i = 0, 1$, so that γ_0, γ_1 are nonseparating curves and $\chi((\Sigma_0)_+) - \chi((\Sigma_0)_-) = \chi((\Sigma_1)_+) - \chi((\Sigma_1)_-)$, where $(\Sigma_i)_+$, $(\Sigma_i)_-$ are the positive and negative regions of $\Sigma_i \backslash \Gamma_{\Sigma_i}$. Then choose a characteristic foliation $\mathcal{F}$ on ∂M which is adapted to $\Gamma_{\Sigma_0} \sqcup \Gamma_{\Sigma_1}$. We have the following:*

(1) *All the tight contact structures which satisfy the boundary condition $\mathcal{F}$ are universally tight.*

(2) *If $\gamma_0 \neq \gamma_1$, then $\#\pi_0(\mathrm{Tight}\,(M, \mathcal{F})) = 4$. Provided γ_0 and γ_1 are not homologous, they are distinguished by the (Poincaré duals to the) relative Euler classes $\mathrm{PD}(\widetilde{e}(\xi)) = \pm\gamma_0 \pm \gamma_1$.*

(3) *If $\gamma_0 = \gamma_1$, then $\#\pi_0(\mathrm{Tight}\,(M, \mathcal{F})) = 5$. Three of them (one of them I-invariant) have relative Euler class $\mathrm{PD}(\widetilde{e}(\xi)) = 0$ and the other two have $\mathrm{PD}(\widetilde{e}(\xi)) = \pm 2\gamma_0 = \pm 2\gamma_1$.*

Here, $\#\pi_0(\mathrm{Tight}\,(M, \mathcal{F}))$ denotes the number of connected components of tight contact 2-plane fields adapted to $\mathcal{F}$, and the *relative Euler class* is an invariant of the tight contact structure which will be defined in Section 4. Theorem 1.1 is a complete classification in the extremal case, provided $\#\Gamma_{\Sigma_i} = 2$ and Γ_{Σ_i} consists of two nonseparating curves. The proof of Theorem 1.1, given in Section 6, requires one involved calculation, found in Section 5, followed by judicious use of general facts from curve complex theory, found in Section 3.

The extremal case is currently the only case we understand, but we also have the following theorem:

Theorem 1.2. *Let M be a closed, oriented, hyperbolic 3-manifold which fibers over S^1, where the fiber is a closed oriented surface Σ of genus $g > 1$ and the monodromy map is pseudo-Anosov. Then there exists a unique tight contact structure up to isotopy in each of the two* extremal cases, *i.e.*, $\langle e(\xi), \Sigma\rangle = \pm(2g-2)$. *This contact structure is universally tight and weakly symplectically fillable. Moreover, every C^0-small perturbation of the fibration into a contact structure is isotopic to the unique extremal tight contact structure.*

Perturbing the fibration by Σ into a contact structure is either done directly or by appealing to the perturbation result of Eliashberg and Thurston [7], which also tells us that the contact structure is weakly symplectically semi-fillable. It is interesting to note that, no matter how we perturb the fibration into a contact structure, the resulting tight contact structure is the same. This contrasts with the case where the fiber is a torus [10].

In this paper we adopt the following conventions:

(1) The ambient manifold is an oriented, compact 3-manifold.

(2) Contact structures are assumed to be co-oriented.

(3) A convex surface S is either closed or compact with Legendrian boundary.

(4) Γ_S = dividing multicurve of a convex surface S.

(5) $\#\Gamma_S$ = number of connected components of Γ_S.

(6) $S \setminus \Gamma_S = S_+ \cup S_-$, where S_+ (resp. S_-) is the region where the normal orientation of S is the same as (resp. opposite to) the normal orientation for ξ.

(7) $|\beta \cap \gamma|$ = geometric intersection number of two curves β and γ on a surface.

(8) $\#(\beta \cap \gamma)$ = cardinality of the intersection.

(9) $S \setminus \gamma$ = metric closure of the complement of γ in S.

(10) $t(\beta, Fr_S)$ = twisting number of a Legendrian curve with respect to the framing induced from the surface S.

2. Tools from convex surface theory

In this section we collect some results from the theory of convex surfaces which are nonstandard. For standard results on convex surfaces, we refer the reader to [9, 12, 14, 17, 18].

We first recall the Legendrian realization principle (LeRP), in a slightly stronger form. An embedded graph C on a convex surface S is *nonisolating* if:

(1) C is transverse to Γ_S,

(2) the univalent vertices of C lie on Γ_S,

(3) all the other vertices do not lie on Γ_S, and

(4) every component of $S\backslash(\Gamma_S \cup C)$ has a boundary component which intersects Γ_S.

Theorem 2.1 (Legendrian realization)**.** *Let C be a nonisolating graph on a convex surface S and v a contact vector field transverse to S. Then there exists an isotopy ϕ_s, $s \in [0,1]$ so that:*

(1) $\phi_0 = \mathrm{id}$, $\phi_s|_{\Gamma_S} = \mathrm{id}$.

(2) $\phi_s(S) \pitchfork v$ (*and hence $\phi_s(S)$ are all convex*),

(3) $\phi_1(\Gamma_S) = \Gamma_{\phi_1(S)}$,

(4) $\phi_1(C)$ *is Legendrian.*

Proof. The proof is identical to the Legendrian realization principle of [14]. q.e.d.

Next, let S be a convex surface in a tight contact manifold (M, ξ) and let δ be a Legendrian arc which is transverse to Γ_S, has endpoints on Γ_S and satisfies $|\Gamma_S \cap \delta| = 3$. We ask whether this Legendrian arc δ is an arc of attachment for a bypass in (M, ξ). In general, this is a difficult question — however, there is one case when there is always a positive answer, namely when δ is a *trivial arc*. We say δ is a *trivial arc* if there exists a disk $D \subset S$ which contains δ in its interior so that $\partial D \pitchfork \Gamma_S$ and an *abstract bypass move* does not change the isotopy type of $\Gamma_S \cap D$ relative to ∂D. Here, by an *abstract bypass move* we simply mean a modification of the multicurve $\Gamma_S \subset S$ in a neighborhood of δ which would theoretically arise from attaching a bypass along δ — there may or may not be an actual bypass half-disk along δ inside the ambient 3-manifold. The following Right-to-Life Principle is an essential tool for analyzing tight contact structures:

Proposition 2.2 (The Right-to-Life Principle)**.** *Let S be a convex surface in a tight contact manifold (M, ξ) and let δ be a trivial Legendrian arc, as described in the previous paragraph. Then there exists an actual bypass half-disk B along δ (from the proper side) contained in the I-invariant neighborhood of S.*

The Right-to-Life Principle is a consequence of Eliashberg's classification of tight contact structures on the 3-ball [6], and is proved in Lemma 1.8 of [16]. Here we recreate the proof.

Proof. Suppose δ intersects Γ_S successively along p_1, p_2, p_3. Since δ gives rise to a trivial abstract bypass move, we may assume that the subarc from p_1 to p_2, together with a subarc of Γ_S, bounds a disk D'. Let $D \subset S$ be a disk which contains $D' \cup \delta$. We may assume ∂D is Legendrian with $tb(\partial D) = -2$ — to do this we apply LeRP (or Giroux's Flexibility Theorem [9, 14]) to ∂D, while fixing δ. Then Γ_D consists of two arcs $\gamma_1 \ni p_1, p_2$ and $\gamma_2 \ni p_3$.

We will now find the actual bypass along $\delta \times \{0\}$ inside the I-invariant tight contact structure on $D \times [0,1]$. (Here we are assuming that the bypass attached from the $D \times [0,1]$-direction is the trivial bypass.) Let p_0 be a point on γ_2 $(\neq p_3)$ for which there exists a Legendrian arc $\delta' \subset D$ from p_1 to p_0 which does not intersect Γ_D except at endpoints and does not intersect δ except at p_1. Define $\delta_0 = \delta \cup \delta'$ and $\delta_1 \subset D$ to be a Legendrian arc from p_3 to p_0 that lies on the same side of γ_2 as δ_0 and has no other intersections with Γ_D. Now consider an arc $\varepsilon \subset D$ from p_0 to p_3 that lies on the opposite side of γ_2 as δ_0 and δ_1 and has no other intersections with Γ_D. Let $A \subset D \times [0,1]$ be a convex annulus such that $\partial A = (\delta_0 \cup \varepsilon) \times \{0\} \cup (\delta_1 \cup \varepsilon) \times \{1\}$ and such that $(\varepsilon \times [0,1]) \subset A$. Since $\Gamma_{\varepsilon \times I}$ consists of an arc $\{q\} \times [0,1]$ $(q \in \varepsilon)$, Γ_A must contain one of two possible bypasses, one intersecting $\{p_0, p_1, p_2\} \times \{0\}$ and the other intersecting $\{p_1, p_2, p_3\} \times \{0\}$. The former is a bypass which cannot exist since it gives rise to an overtwisted disk. Therefore we obtain the second, as stated in the proposition. q.e.d.

Lemma 2.3 (Bypass Sliding Lemma)**.** *Let R be an embedded rectangle with consecutive sides a, b, c, d in a convex surface S such that a is an arc of attachment of a bypass, b and d are subsets of Γ_S and c is a Legendrian arc which is efficient (rel endpoints) with respect to Γ_S. Then there exists a bypass for which c is its arc of attachment.*

Proof. The appendix to [15] explains how to explicitly move the endpoints of the Legendrian arc of attachment. In this paper, we will deduce the Bypass Sliding Lemma from the Right-to-Life Principle.

Let $R' \supset R$ be an embedded disk in S satisfying the following:

- $\partial R'$ is Legendrian, $\partial R' \pitchfork \Gamma_S$, and $tb(\partial R') = -3$.
- $\partial R' = a' \cup b' \cup c' \cup d'$, where a' and c' are Legendrian arcs parallel to and close to a and c, respectively. The four arcs a', b', c', d' are consecutive sides of a rectangle whose corners, lying on Γ_S, have been smoothed.
- $\Gamma_{R'}$ consists of three parallel (= nested) dividing arcs.

Such an R' exists by LeRP. Suppose we attach the bypass along a to obtain a convex surface R'' isotopic to R' relative to $\partial R'$. Note that, away from a, R'' and R' are identical. Now, the abstract bypass move applied to R'' along c still yields the same dividing set $\Gamma_{R''}$. Therefore, the bypass must also exist along c. (In fact, it exists in a small neighborhood of R''.) q.e.d.

Lemma 2.4. *Let S be a closed convex surface of genus g, with dividing set Γ_S which consists of one homotopically nontrivial separating curve. If $S \times I$ is the I-invariant neighborhood of $S = S \times \{\frac{1}{2}\}$, then there exists a bypass B in $S \times I$ along $S \times \{1\}$ such that the convex surface obtained by isotoping $S \times \{1\}$ across B has dividing set consisting of three curves parallel to Γ_S.*

The arc of attachment δ for the bypass B is contained in an annular neighborhood of Γ_S and intersects Γ_S in 3 points, and the two half-disks bounded by δ and Γ_S have a common intersection along an arc contained in Γ_S. See the right-hand diagram in Figure 1.

For a further discussion on dividing-curve increasing bypasses, see [16].

Remarks.

(1) Typically, we increase $\#\Gamma$ by *folding* along a Legendrian divide (see [14] or [18]). This generates a pair of dividing curves parallel to the Legendrian divide. However, this folding operation to increase $\#\Gamma$ does not occur immediately for free, in case the curve we want to realize as a Legendrian divide is an *isolating* curve for Γ in the sense of LeRP. This is the case in Lemma 2.4. Since

Lemma 2.4 (or the proof therein) shows that folds along homotopically nontrivial closed curves always exist, Lemma 2.4 can be viewed as a strengthened form of LeRP and of Giroux's Flexibility Theorem.

(2) In [18], it was explained that unfolding (the reverse procedure of folding) is equivalent to adding a "short" bypass and, conversely, folding is equivalent to digging (attaching from the other side) the dual "long" bypass. See Figure 1 for illustrations of "long" and "short" bypasses.

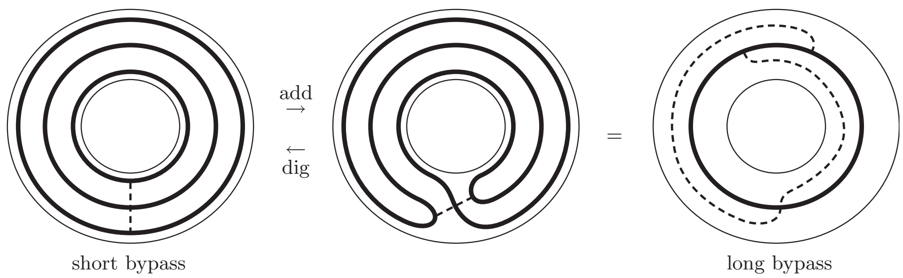

Figure 1: Folding and unfolding in terms of bypass addition and digging.

Proof. The statement of the Flexibility Theorem involves a choice of contact vector field $v \pitchfork S$ which defines the product structure $S \times I$. To change the foliation on S, S is moved via an *admissible isotopy*, i.e., an isotopy ϕ_s, $s \in [0,1]$, such that $\phi_s(S) \pitchfork v$ for all s. Also recall that we may take $\phi_s|_{\Gamma_S} = \mathrm{id}$ and also leave certain pre-specified Legendrian arcs with boundary on Γ_S invariant under ϕ_s. The dividing set of S clearly remains unchanged under an admissible isotopy.

Let $\delta \subset S$ be a Legendrian arc so that $\delta \times \{1\}$ is the Legendrian arc of attachment for the bypass we are looking for. Let $\beta \subset S - \Gamma_S$ be a nonseparating curve and $\gamma \subset S - \Gamma_S$ be a parallel push-off of Γ_S in the same connected component as β. Assume that δ, β, and γ are disjoint. Since we cannot immediately apply LeRP to γ, we first apply LeRP to β, fold along β to create two new dividing curves parallel to β, and then apply LeRP to γ. Call this convex surface S'. We then fold along the Legendrian-realized γ to construct a convex surface S'' isotopic to S that has two extra dividing curves parallel to β and two extra dividing curves parallel to γ. Now, by Remark (2) above, the folding operation from S' to S'' is equivalent to attaching a long bypass B along $\delta' \subset S'$ as in Figure 1. Corresponding to the two types of folds along γ, there are long bypasses from either side of S' — take the long bypass away from $S \times \{1\}$.

The key point here is that the isotopy from S to S' can be taken to be supported away from $\Gamma_S \cup \delta$. This is because the admissible isotopy in LeRP can be taken to be constant on $\Gamma_S \cup \delta$, and the folding operation is local near the Legendrian-realized β. Therefore, we may take $\delta' = \delta$, and the bypass we are looking for is $B \cup (\delta \times [\frac{1}{2}, 1])$. q.e.d.

3. Tools from curve complex theory

In this section, we recall several facts from the theory of curve complexes which will become useful later. Let Σ be a closed oriented surface of genus $g \geq 2$ — we stress here that all the lemmas and propositions of this section require $g \geq 2$. Let $\mathcal{C}(\Sigma)$ be the curve complex for Σ. The vertices of $\mathcal{C}(\Sigma)$ consist of isotopy classes of homotopically nontrivial closed curves on Σ and there is a single edge between each pair γ_0, γ_1 of (isotopy classes of) closed curves with $|\gamma_0 \cap \gamma_1| = 0$. Note that we do not attach simplices of higher dimension, since they are not needed. The facts we need from the theory of curve complexes are variations and strengthenings of the basic Lemma 3.1 below. We refer the reader to [19] for an exposition on curve complexes, as well as an extensive reference section and a proof of Lemma 3.1.

Lemma 3.1. *$\mathcal{C}(\Sigma)$ is connected, i.e., given two closed curves α, α' in $\mathcal{C}(\Sigma)$, there exists a sequence $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_k = \alpha'$ in $\mathcal{C}(\Sigma)$ where $|\alpha_{i-1} \cap \alpha_i| = 0$, $i = 1, \ldots, k$.*

Alternatively, we have the following:

Lemma 3.2. *Given two nonseparating closed curves α, α' on Σ, there exists a sequence $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_k = \alpha'$ of closed curves where $|\alpha_{i-1} \cap \alpha_i| = 1$, $i = 1, \ldots, k$.*

Proposition 3.3. *Given two nonseparating closed curves α and α' on Σ, there exists a sequence $\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_k = \alpha'$ of closed curves where:*

(1) *α_i are nonseparating, $i = 0, \ldots, k$.*

(2) *$|\alpha_{i-1} \cap \alpha_i| = 0$, $i = 1, \ldots, k$.*

(3) *α_{i-1} and α_i are not homologically equivalent, $i = 1, \ldots, k$.*

Proof. By Lemma 3.2, there is a sequence $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_k = \alpha'$ of closed curves where $|\alpha_{i-1} \cap \alpha_i| = 1$, $i = 1, \dots, k$. A neighborhood U_{i-1} of $\alpha_{i-1} \cup \alpha_i$ is a punctured torus. Since the genus of Σ is at least 2, $\Sigma - U_{i-1}$ contains a nonseparating curve β_{i-1}. Therefore, $\alpha_0, \beta_0, \alpha_1, \beta_1, \dots, \alpha_k$ is the desired sequence. q.e.d.

Proposition 3.4. *Given three nonseparating closed curves α, β, β' on Σ, where $|\alpha \cap \beta| = 0$, $|\alpha \cap \beta'| = 0$, α and β are nonhomologous, and α and β' are nonhomologous, there exists a sequence $\beta_0 = \beta, \beta_1, \dots, \beta_k = \beta'$ of closed curves where:*

(1) *β_i are nonseparating, $i = 0, \dots, k$.*

(2) *$|\alpha \cap \beta_i| = 0$, $i = 0, \dots, k$.*

(3) *$|\beta_{i-1} \cap \beta_i| = 1$, $i = 1, \dots, k$.*

Proof. By assumption $\Sigma \setminus \alpha$ is connected. If γ is a curve in Σ such that $|\gamma \cap \alpha| = 0$, then γ is nonseparating and not homologous to α if and only if γ is nonseparating as a subset of $\Sigma \setminus \alpha$. Let Σ' be $\Sigma \setminus \alpha$ together with two disks capping off the boundary components corresponding to α. Then β and β' are nonseparating closed curves of Σ' and applying Lemma 3.2 to β, β' and Σ' produces the desired sequence of curves. q.e.d.

Proposition 3.5. *Given two nonseparating closed curves α and α' on Σ, there exists a sequence $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_k = \alpha'$ of closed curves where:*

(1) *α_i are nonseparating, $i = 0, \dots k$.*

(2) *$|\alpha_{i-1} \cap \alpha_i| = 1$, $i = 1, \dots, k$.*

(3) *$|\alpha_{i-1} \cap \alpha_{i+1}| = 0$, $i = 1, \dots, k-1$.*

Proof. Let $\alpha_0 = \alpha, \alpha_1, \dots, \alpha_k = \alpha'$ be a sequence for which $|\alpha_{i-1} \cap \alpha_i| = 1$, $i = 1, \dots, k$, as guaranteed by Lemma 3.2. Suppose that $j \geq 1$ is the smallest integer for which $|\alpha_{j-1} \cap \alpha_{j+1}| > 0$. We show how to insert curves into the sequence between α_j and α_{j+1}, until Condition 3 is satisfied, at least up to α_{j+1}. Since curves are only inserted after α_j, there is no danger that Condition 3 will be disturbed for earlier terms in the sequence.

For notational convenience, we shall take $j = 1$. We will always assume that $\alpha_0 \cap \alpha_1$ is not an element of α_2. Suppose that $|\alpha_0 \cap \alpha_2| \geq 1$.

The following cases are used to inductively decrease the number of intersections between α_0 and α_2.

Case 1. Not all intersections between α_0 and α_2 have the same sign.

Then there exist two points $p, q \in \alpha_0 \cap \alpha_2$ with opposite signs of intersection, and an arc $[p, q]_{\alpha_0} \subset \alpha_0$ which connects p and q such that $[p, q]_{\alpha_0}$ does not intersect α_1 and contains no point of α_2 in its interior. Cut-and-paste surgery of α_2 along $[p, q]_{\alpha_0}$ produces two curves; let β the component which intersects α_1 once. The sequence $\alpha_0, \alpha_1, \beta, \alpha_1, \alpha_2$ satisfies Condition 2 (and hence Condition 1), and is closer to satisfying Condition 3 than the original sequence, in the sense that the total number of intersection points of type $\alpha_{i-1} \cap \alpha_{i+1}$ has been reduced. Here it is understood that the second α_1 in the sequence is slightly isotoped off the first α_1 so that the two curves are disjoint as required in Condition 3.

Case 2. All intersections of α_0 and α_2 have the same sign and $|\alpha_0 \cap \alpha_2| \geq 2$.

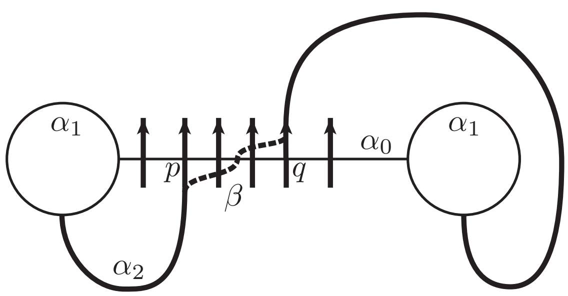

Figure 2: Case 2.

Let $[p, q]_{\alpha_2}$, $p, q \in \alpha_0 \cap \alpha_2$, be the smallest interval in α_2 that contains $\alpha_1 \cap \alpha_2$. Let $[p, q]_{\alpha_0}$ be the interval in α_0 (with endpoints p, q) which does not intersect α_1. Let $\beta = [p, q]_{\alpha_2} \cup [p, q]_{\alpha_0}$. After a slight isotopy of β, the sequence $\alpha_0, \alpha_1, \beta, \alpha_1, \alpha_2$ satisfies Condition 2. Condition 3 is not satisfied, but at least the first and third terms of this interim sequence intersect only once and $|\beta \cap \alpha_2| < |\alpha_0 \cap \alpha_2|$.

Case 3. $|\alpha_0 \cap \alpha_2| = 1$.

Let U be a regular neighborhood of $\alpha_0 \cup \alpha_1 \cup \alpha_2$ in Σ. Let $V = U \setminus (\alpha_0 \cup \alpha_1)$. V is a planar surface with 4 boundary components, one corresponding to $\alpha_0 \cup \alpha_1$, shown as a rectangle in Figure 3, and three others denoted A, B and C. The construction of the desired sequence of curves depends on the relation of A, B and C to the rest of Σ. In each of the following cases, S will denote a connected subsurface of Σ which satisfies $\partial S \subset A \cup B \cup C$ and whose interior $\mathrm{int}(S)$ is disjoint from V.

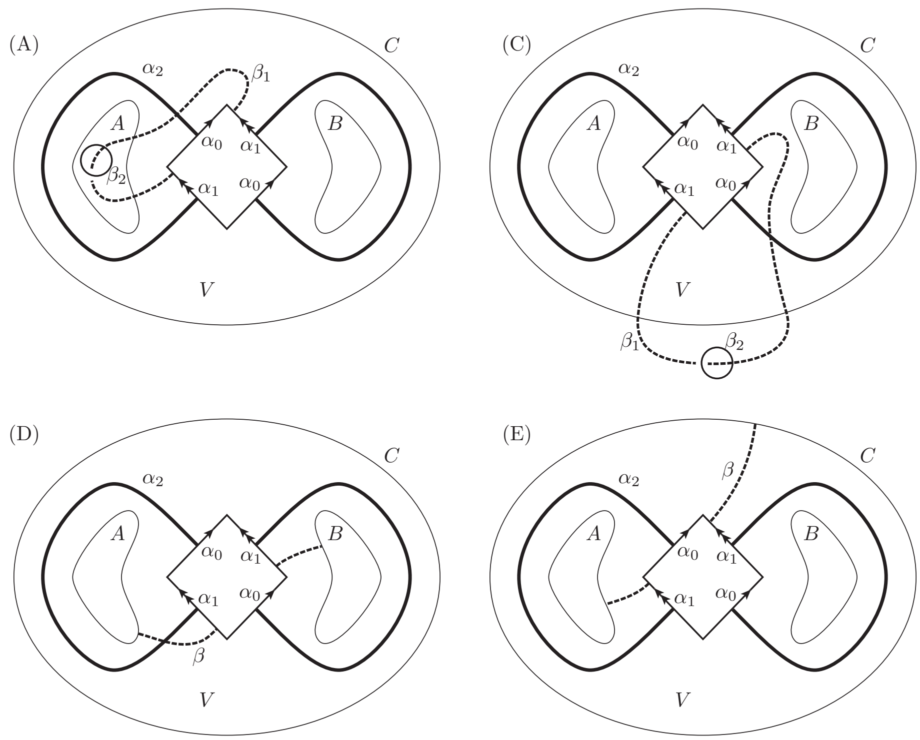

Figure 3: Case 3.

(A) $A \subset S$ and genus$(S) \geq 1$. Choose β_1 as indicated in Figure 3(A), that is, β_1 starts on α_1, then passes across A into S and back to A without separating S, and then crosses α_2 before returning to the point it started from, but from the opposite side of α_1. Let $\beta_2 \subset S$ be a curve such that $|\beta_2 \cap \beta_1| = 1$. The sequence $\alpha_0, \alpha_1, \beta_1, \beta_2, \beta_1, \alpha_2$ satisfies Conditions 1–3.

(B) $B \subset S$ and genus$(S) \geq 1$. This is similar to (A).

(C) $C \subset S$ and genus$(S) \geq 1$. Choose β_1 and β_2 as shown in Figure 3(C) and the sequence $\alpha_0, \alpha_1, \beta_1, \beta_2, \beta_1, \alpha_2$ will satisfy Conditions 1–3.

(D) $A \cup B \subset S$. Choose an arc in V that runs from A, across α_2 and α_1 and then to B. Complete this arc to a curve β by attaching an arc in S. The sequence $\alpha_0, \alpha_1, \beta, A, \beta, \alpha_2$ satisfies Conditions 1–3.

(E) $A \cup C \subset S$. Choose an arc in V that runs from C, across α_1 and then to A. Complete this arc to a curve β by attaching an arc in S. The sequence $\alpha_0, \alpha_1, \beta, \alpha_1, \alpha_2$ satisfies Conditions 1–3.

(F) $B \cup C \subset S$. This is similar to (E).

The only possibility not covered in (A)–(F) is if each of A, B and C bound disks, but this would imply genus$(\Sigma) = 1$, contrary to assumption. q.e.d.

4. Definition of the relative Euler class

Let $M = \Sigma \times I = \Sigma \times [0,1]$ and $\Sigma_t = \Sigma \times \{t\}$. If Σ_t is convex, we denote $\Gamma_t = \Gamma_{\Sigma_t}$. Let ξ be a tight contact structure on M which satisfies $\langle e(\xi), \Sigma_t \rangle = \pm(2g-2)$. Since the situation is symmetric under sign change, we will additionally assume that $\langle e(\xi), \Sigma_t \rangle = -(2g-2)$.

First recall the following identity (cf. Kanda [21] or Eliashberg [6]): If S is a closed convex surface, then

$$\langle e(\xi), S \rangle = \chi(S_+) - \chi(S_-). \tag{1}$$

According to Giroux's criterion [12], a closed convex surface $S \neq S^2$ has a tight neighborhood if and only if no connected component of $S_\pm$ is a disk. For a convex fiber Σ of (M, ξ), we have $\chi(\Sigma_\pm) \leq 0$. Hence, the extremal condition implies that $\chi(\Sigma_+) = -(2g-2)$ and $\chi(\Sigma_-) = 0$. In other words, Σ_- is a nonempty union of annuli. Here, recall that both Σ_+ and Σ_- of a convex surface Σ are nonempty, since there must be both sources (Σ_+) and sinks (Σ_-) for the characteristic foliation.

This section is devoted to defining the *relative Euler class* $\widetilde{e}(\xi) \in H^2(M, \partial M; \mathbb{Z})$, a natural extension of the Euler class $e(\xi) \in H^2(M; \mathbb{Z})$ when $\langle e(\xi), \Sigma \rangle = \pm(2g-2)$. We define the relative Euler class $\widetilde{e}(\xi)$ as a map $H_2(M, \partial M; \mathbb{Z}) \to \mathbb{Z}$ by evaluating on annuli, or equivalently as a map $H_1(\Sigma; \mathbb{Z}) \to \mathbb{Z}$, by evaluating on closed curves. Let γ be a closed curve on Σ. If $\gamma \times \{0,1\}$ is *efficient* with respect to ∂M (i.e., intersects $\Gamma_{\partial M}$ minimally in its isotopy class in ∂M) and is *nonisolating* in ∂M (which is the case for example when γ is nonseparating in Σ), then we may use the Legendrian realization principle (LeRP) [14] to make $\gamma \times \{0,1\}$ Legendrian. If $S \subset M$ is a convex annulus with boundary $\gamma \times \{0,1\}$, then we define:

$$\langle \widetilde{e}(\xi), S \rangle = \chi(S_+) - \chi(S_-). \tag{2}$$

In general, choose a suitable representative $\gamma \times \{0,1\}$ in its isotopy class, i.e., introduce extra intersections with $\Gamma_{\partial M}$, so that the following condition holds:

$\gamma \times \{0,1\}$ *is nonisolating in* $\Sigma \times \{0,1\}$*, and every connected component of* $(\gamma \times \{i\}) \cap (\Sigma_i)_-$*,* $i = 0, 1$*, is a nonseparating arc in* $(\Sigma_i)_-$*.*

Such a $\gamma \times \{0,1\}$ will be called *primped*. Observe that (i) $(\Sigma_i)_-$ is a union of annuli, and (ii) components of $(\gamma \times \{i\}) \cap (\Sigma_i)_+$ may be separating in $(\Sigma_i)_+$. (In other words, extra intersections are introduced by performing a finger move across annular components of $(\Sigma_i)_-$.) If S is a convex annulus with primped $\partial S = \gamma \times \{0,1\}$, we define $\langle \widetilde{e}(\xi), S \rangle$ as in Equation (2).

Proposition 4.1. *The relative Euler class* $\widetilde{e}(\xi)$ *is a well-defined element in* $H^2(M, \partial M; \mathbb{Z})$*.*

Proof. The following lemma proves that $\widetilde{e}(\xi)$ gives a well-defined map from the isotopy classes of oriented closed curves on Σ to $\mathbb{Z}$.

Lemma 4.2. *Let* S *and* S' *be isotopic convex annuli in* M *with primped Legendrian boundary* $\delta \times \{0,1\}$ *and* $\delta' \times \{0,1\}$*. Then* $\langle \widetilde{e}(\xi), S \rangle = \langle \widetilde{e}(\xi), S' \rangle$*.*

Proof of Lemma 4.2. First consider the case where $\partial S = \partial S'$. After perturbing S and S' slightly near the boundary and rounding along the common edge, we may take $S \cup (-S')$ to be a closed immersed torus. Although Equation (1) is stated only for closed convex surfaces (convex surfaces are embedded by definition), Kanda's argument in [21] also applies to immersed surfaces which have meaningful positive and negative regions; this is because the quantities involved are homotopy-theoretic. Therefore, we have:

$$\begin{aligned} \langle \widetilde{e}(\xi), S \rangle - \langle \widetilde{e}(\xi), S' \rangle &= [\chi(S_+) - \chi(S_-)] - [\chi(S'_+) - \chi(S'_-)] \\ &= \chi(S_+ \cup S'_-) - \chi(S_- \cup S'_+) \\ &= \langle e(\xi), S \cup (-S') \rangle = 0. \end{aligned}$$

Hence $\langle \widetilde{e}(\xi), S \rangle$ is independent of the choice of S, provided ∂S is fixed.

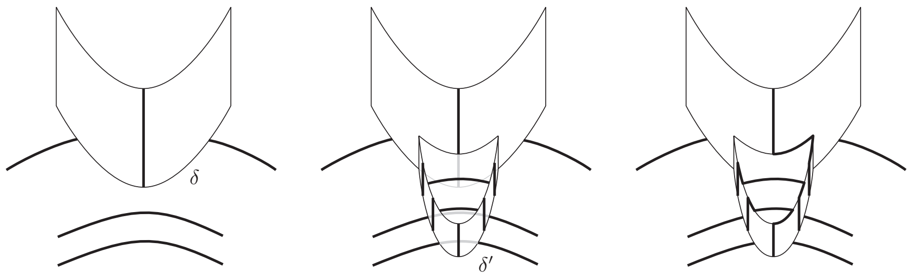

Figure 4: Isotopy of δ to δ'.

Next consider the situation where $\partial S \neq \partial S'$. The key point which makes the whole argument work is that dividing curves of Σ_i come in pairs *in the extremal case*. Consequently, two primped curves δ and δ' in the same isotopy class can be isotoped into one another through a sequence of operations (depicted in Figure 4), where δ is pushed simultaneously across two parallel curves of Γ_{Σ_i}. Such an isotopy changes the dividing set on the corresponding annulus S by adding or deleting a "nested pair of ∂-parallel arcs", i.e., two parallel arcs, one of which is ∂-parallel. Since a nested pair consists of one disk region each for S_+ and S_-, $\langle \widetilde{e}(\xi), S\rangle$ is invariant under this isotopy. After a finite sequence of such steps, we are in the case where $\partial S = \partial S'$. q.e.d.

Next, we prove additivity.

Lemma 4.3. *Let δ_i, $i = 1, 2$, be two oriented multicurves on Σ, where $\delta_1 \pitchfork \delta_2$ and $(\delta_1 \cap \delta_2) \cap (\Gamma_\Sigma \cup \Sigma_-) = \emptyset$, and let δ be the oriented multicurve obtained by resolving their intersections.* (*Hence* $[\delta_1] + [\delta_2] = [\delta] \in H_1(\Sigma)$.) *If S_i is isotopic to $\delta_i \times I$ and S to $\delta \times I$, then*

$$\langle \widetilde{e}(\xi), S\rangle = \langle \widetilde{e}(\xi), S_1\rangle + \langle \widetilde{e}(\xi), S_2\rangle.$$

Proof of Lemma 4.3. Consider the graph $C = \delta_1 \cup \delta_2$. By Lemma 4.2, we may assume without loss of generality that enough extra intersections of $\Gamma_\Sigma \cap \delta_i$ have been introduced to δ_i, $i = 1, 2$, so that δ_i is primped and C is nonisolating. Applying a slightly stronger version of LeRP (easily derived from Giroux's Flexibility Theorem [14]), we may take C to be a Legendrian graph with elliptic tangencies $\delta_1 \cap \delta_2$. By perturbing if necessary, we take $(\delta_1 \cap \delta_2) \times I$ to be transverse curves. If we cut and paste along $(\delta_1 \cap \delta_2) \times I$ and smooth the corners in the proper manner, we obtain the surface $S = \delta \times I$ with Legendrian boundary, satisfying the desired equality. The equality follows from relating $\chi(S_+) - \chi(S_-)$ to the standard way of computing $\langle \widetilde{e}(\xi), S\rangle$ using the sign and type of isolated singularities (for example see Kanda's argument in [21]). Observe that ∂S is primped, since $\delta_1 \cap \delta_2$ were chosen to be away from Σ_-. q.e.d.

It now remains to show the following:

Lemma 4.4. *Let δ_i, $i = 1, 2$, be two oriented multicurves representing the same homology class in $H_1(\Sigma)$. Then $\langle \widetilde{e}(\xi), \delta_1 \times I\rangle = \langle \widetilde{e}(\xi), \delta_2 \times I\rangle$.*

Proof of Lemma 4.4. Let $\Sigma' \subset \Sigma$ be a subsurface. Then we compute that

$$\langle \widetilde{e}(\xi), \partial\Sigma' \times I\rangle = 0.$$

Indeed, this follows from

$$\begin{aligned} 0 = \langle e(\xi), \partial(\Sigma' \times I)\rangle &= \chi^{\#}(\partial(\Sigma' \times I)) \\ &= \chi^{\#}(\Sigma' \times \{1\}) - \chi^{\#}(\Sigma' \times \{0\}) + \chi^{\#}((\partial\Sigma') \times I), \end{aligned}$$

coupled with the observation that if $\partial\Sigma' \times \{0, 1\}$ is primped, then

$$\chi^{\#}(\Sigma' \times \{0\}) = \chi^{\#}(\Sigma' \times \{1\}) = \chi(\Sigma').$$

Here, for convenience we write $\chi^{\#}(\cdot) = \chi((\cdot)_+) - \chi((\cdot)_-)$.

Now, if δ_1 and δ_2 are oriented multicurves in the same homology class, then we may assume after isotopy that δ_i satisfy the conditions of Lemma 4.3. Let δ be the resolution of $\delta_1 - \delta_2$ (note that δ is embedded). In view of Lemma 4.3, it suffices to show that $\langle \widetilde{e}(\xi), \delta \times I\rangle = 0$. Separating curves of δ (including homotopically trivial curves) can be thrown away by the previous paragraph, while pairs of parallel curves with opposite orientation can also be removed. Therefore, we may represent δ as $\sum_i m_i\gamma_i$, where m_i are positive integers and γ_i are oriented nonseparating curves. If $\Sigma \setminus (\cup_i\gamma_i)$ is connected, then there is a closed curve which intersects some γ_i exactly once, contradicting the assumption of homological triviality of δ. Now, supposing $\Sigma \setminus (\cup_i\gamma_i)$ is not connected, we may take a suitable (possibly negative) multiple of a connected component Σ' so that $\sum_i m_i\gamma_i + k\partial\Sigma'$ has fewer terms in the sum. Such a modification is possible since $(\partial\Sigma') \times I$ does not contribute to the relative Euler class. By decreasing the number of summands in δ, we eventually prove that $\langle \widetilde{e}(\xi), \delta \times I\rangle = 0$. q.e.d.

This completes the proof of Proposition 4.1. q.e.d.

The relative Euler class $\widetilde{e}(\xi) \in H^2(M, \partial M; \mathbb{Z})$ will often be expressed in terms of its Poincaré dual $\mathrm{PD}(\widetilde{e}(\xi)) \in H_1(M; \mathbb{Z}) \simeq H_1(\Sigma; \mathbb{Z})$.

5. Computation of the base case

Recall that, by our choice of $\langle e(\xi), \Sigma_t\rangle$, $(\Sigma_i)_-$, $i = 0, 1$, is a union of annuli. If we assume that there is only one pair of dividing curves on each Σ_i, then $(\Sigma_i)_-$ is an annulus and $(\Sigma_i)_+$ is its complement in Σ_i (i.e., "most" of the surface).

We will now consider the following special case, which turns out to be the most fundamental:

Base Case. $\Gamma_i = 2\gamma_i$, $i = 0, 1$, and $|\gamma_0 \cap \gamma_1| = 1$.

Here, $k\gamma$ is shorthand for k parallel (mutually nonintersecting) copies of a closed curve γ.

Theorem 5.1. *Let Σ be a closed surface of genus $g \geq 2$, $M = \Sigma \times I$ the ambient manifold, and $\mathcal{F}$ a characteristic foliation on ∂M adapted to $\Gamma_0 \sqcup \Gamma_1$ of the Base Case. Let* $\mathrm{Tight}\,(M, \mathcal{F})$ *be the space of tight contact 2-plane fields which induce $\mathcal{F}$ along ∂M. Then we have the following:*

(1) $\#\pi_0(\mathrm{Tight}\,(M, \mathcal{F})) = 4$.

(2) *The isotopy classes of tight contact structures are distinguished by the Poincaré duals to their relative Euler class, which are:*

$$\mathrm{PD}(\widetilde{e}(\xi)) = \pm\gamma_0 \pm \gamma_1 \in H_1(M; \mathbb{Z}).$$

(3) *All the tight contact structures are universally tight.*

This computation is a little involved, and occupies Sections 5.1 through 5.4. In what follows, when we prescribe a boundary condition for a 3-manifold M, we will simply give $\Gamma_{\partial M}$, although, strictly speaking, we need to also assign the characteristic foliation $\mathcal{F}$ on ∂M *adapted to* $\Gamma_{\partial M}$. We will assume that some convenient characteristic foliation is prescribed, since the number of tight contact structures is independent of the actual characteristic foliation adapted to $\Gamma_{\partial M}$ (see [14]). The following is a preliminary lemma.

Lemma 5.2. *There exists a unique tight contact structure on $H = S \times I$, where S is a compact oriented surface with nonempty boundary, S_0, S_1, and $(\partial S)\times I$ are convex, $\Gamma_{S_0} = \Gamma_{S_1}$ are ∂-parallel, and $\Gamma_{\partial S\times I}$ are vertical.* (*Here vertical means the dividing curves are arcs $\{pt\}\times I$.*) *This tight contact structure is universally tight, and is obtained by perturbing the foliation of H by leaves $S \times \{t\}$, $t \in [0, 1]$, into a contact structure.*

Proof. Observe that $H = S \times I$ is a handlebody and that $\Gamma_{\partial H}$, after edge-rounding [14], is isotopic to $\partial S \times \{\frac{1}{2}\}$. See Figure 5.

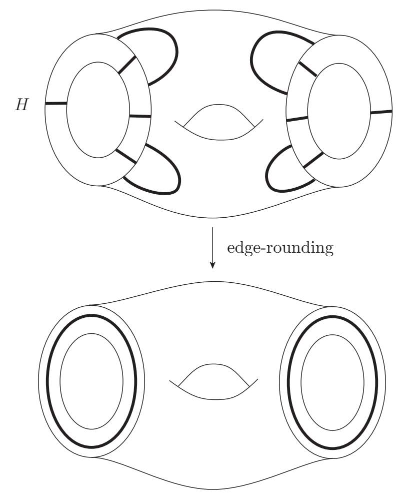

Figure 5: Dividing multicurve after edge-rounding.

Consider a meridional disk $D = \gamma \times I$, where γ is a non-boundary-parallel, properly embedded arc on S. Then, after rounding, ∂D intersects $\Gamma_{\partial H}$ along exactly two points. Make ∂D Legendrian and D convex. Then the Thurston-Bennequin invariant $tb(\partial D)$ equals -1, and there is a unique dividing curve configuration on D consistent with this boundary condition. Moreover, there is a system of such meridional disks $D_1, \dots, D_k$ which decompose H into 3-balls B^3, each with $\Gamma_{\partial B^3} = S^1$. By Eliashberg's uniqueness theorem for tight contact structures on the 3-ball (cf. [6]), there is a unique tight contact structure up to isotopy rel boundary on each of the B^3's. This implies that there exists at most one tight contact structure on H with given $\Gamma_{\partial H}$. Now, to prove that there indeed exists a (universally) tight contact structure on H with given $\Gamma_{\partial H}$, we glue back using Theorem 5.3 below. Note that Γ_{D_i} is ∂-parallel on each meridional disk D_i. We leave the statement of the perturbation to the reader. q.e.d.

Theorem 5.3 (Colin [3]). *Let (M, ξ) be an oriented, compact, connected, irreducible, contact 3-manifold and $S \subset M$ an incompressible convex surface with nonempty Legendrian boundary and ∂-parallel dividing set Γ_S. If $(M \setminus S, \xi|_{M \setminus S})$ is universally tight, then (M, ξ) is universally tight.*

Proof. See [17] for a proof of the theorem stated in this form. q.e.d.

We now begin the analysis of the Base Case. Let us position the dividing curves $2\gamma_0$ and $2\gamma_1$ as in Figure 6, that is, we suppose $\#(\gamma_0 \cap \gamma_1) = 1$ and we have oriented γ_0 and γ_1 so that the intersection pairing $\langle \gamma_1, \gamma_0 \rangle$ on Σ equals $+1$. Now consider an oriented closed curve γ contained in a neighborhood of $\gamma_0 \cup \gamma_1$ which satisfies $\#(\gamma \cap \gamma_i) = 1$ and $\langle \gamma, \gamma_i \rangle = 1$, $i = 0, 1$. After an isotopy of γ_0 and γ_1, we may assume that γ_0 and γ_1 are identical except on a thin annulus $\mathcal{A}$ parallel to but disjoint from γ, obtained as follows. First push γ off of itself in the direction opposite the direction given by the orientation of γ_1 to obtain γ'. Then let $\mathcal{A}$ be a small annular neighborhood of γ'. γ_0 is then obtained from γ_1 via a Dehn twist in $\mathcal{A}$. We will write $\Gamma_{\Sigma_1} \cap \gamma = \{p_1, p_2\} \times \{1\}$, $\Gamma_{\Sigma_0} \cap \gamma = \{p_1, p_2\} \times \{0\}$. Our first cut of $M = \Sigma \times I$ will be along the convex surface $A = \gamma \times I$, where $A = \gamma \times I$ is given the orientation induced from γ and I.

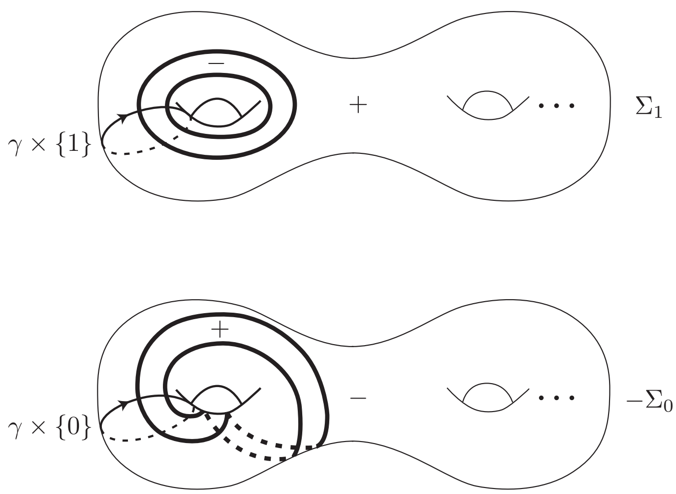

Figure 6: Suitable choice of Γ_0 and Γ_1.

There are two general classes of dividing curves Γ_A which we denote by I_k and $II_n^{\pm}$. The dividing set I_k consists of 2 parallel nonseparating dividing curves (i.e., dividing curves which go across from $\gamma \times \{0\}$ to $\gamma \times \{1\}$). Here, $k \in \mathbb{Z}$ denotes the *holonomy*, or the amount of spiraling, defined as follows. First, *zero holonomy* $k = 0$ means the dividing curves are *vertical* in the sense that they are isotopic rel boundary to $\{q_1, q_2\} \times [0,1]$, where $\{q_1, q_2\} \times \{0,1\}$ are the endpoints of Γ_A. The holonomy is k if Γ_A is obtained from $\{q_1, q_2\} \times [0,1]$ by doing k negative Dehn twists along the core curve of A. The dividing set II_n^+ (resp. II_n^-), $n \in \mathbb{Z}^{\geq 0}$, consists of two ∂-parallel dividing curves which split off half-disks, where the half-disk along $\gamma \times \{1\}$ is positive (resp. negative) and there are n parallel homotopically essential closed curves. See Figure 7 for the possibilities.

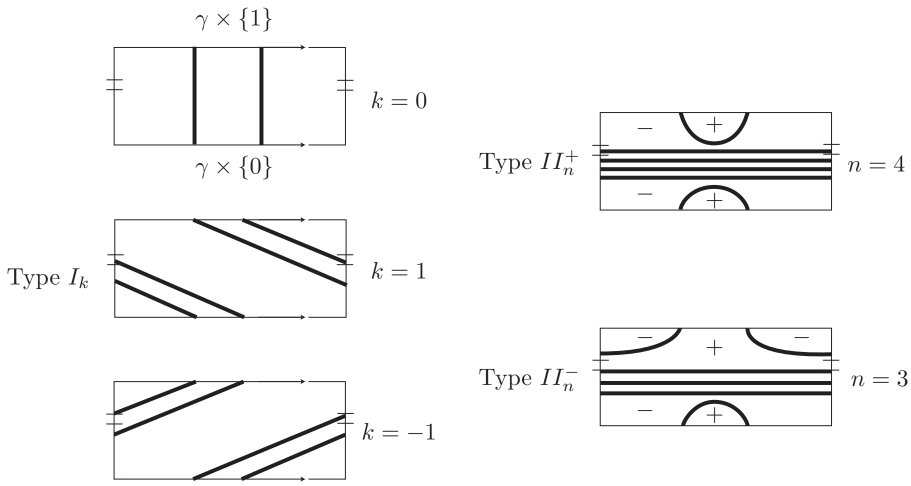

Figure 7: Dividing curves of Type I_k and $II_n^{\pm}$.

5.1 Type $II_n^{\pm}$, n even.

The first case we treat is $\Gamma_A = II_n^{\pm}$, where n is even.

Lemma 5.4. *$II_{2m}^{\pm}$, $m \in \mathbb{Z}^+$, can be reduced to $II_0^{\pm}$.*

Proof. The proof strategy is to start with the convex annulus A with dividing set Γ_A and then search for a bypass B attached along A such that isotoping A across B will produce a convex annulus A' with Γ'_A of decreased complexity. It is important to keep in mind that Theorem 5.1 is false for tori; thus in searching for B we must exploit the assumption that the genus of Σ is ≥ 2.

Assume we have II_{2m}^+. Figure 8 depicts the convex decomposition sequence for this case. We will treat the case $m = 1$, which is the hardest case. The situation $m > 1$ will be left to the reader.

Figure 8(A) depicts M cut open along A. Figure 8(A) shows $(\Sigma \setminus \gamma) \times \{1\}$ together with A^+ and A^-, two copies of A on $M \setminus A$. (Warning:

A^+ and A_+ are distinct surfaces. $A_+ \subset A$ is reserved for the positive region of a convex annulus A, whereas A^+ is the copy of A in $M \setminus A$ where the induced orientation from A agrees with the the orientation induced from $\partial(M \setminus A)$.) Figure 8(B) depicts $(\Sigma \setminus \gamma) \times \{0\}$.

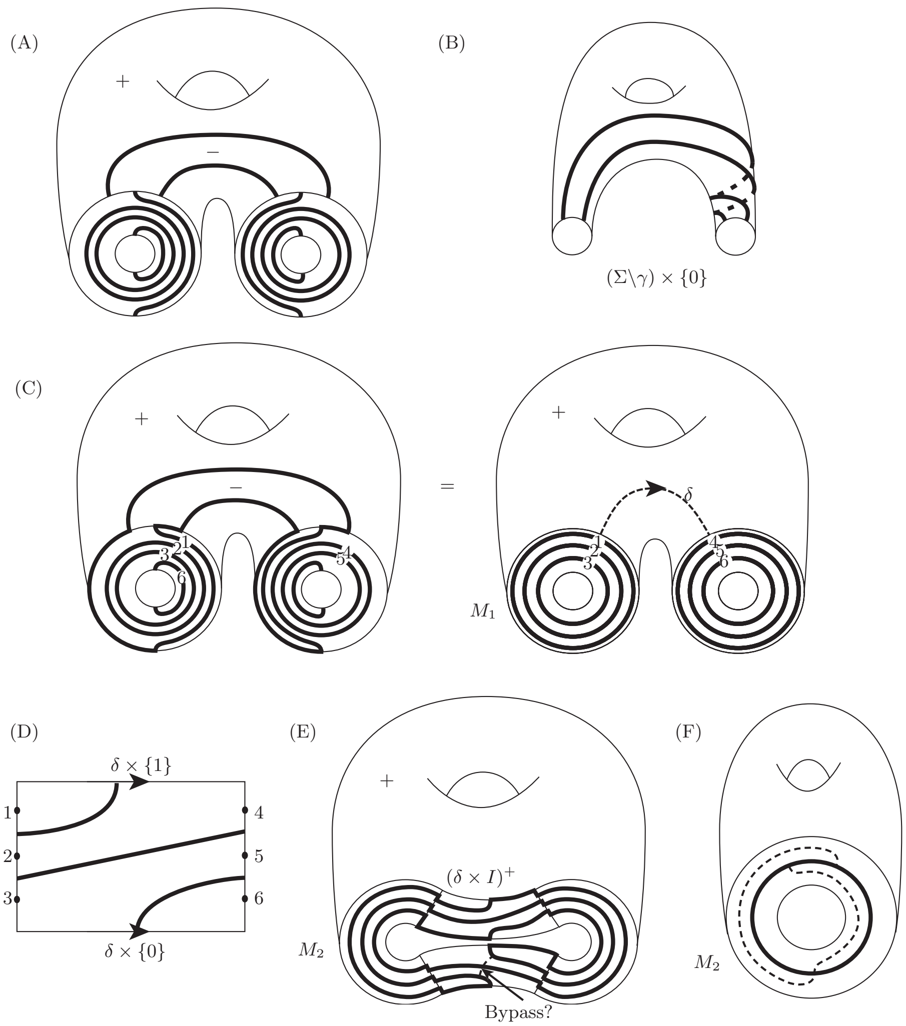

Figure 8: The case II^+_{2m}.

After rounding the edges of $M \setminus A$, we obtain the convex handlebody M_1 in Figure 8(C). Rounding is shown on the left side of Figure 8(C). The right side shows the resulting dividing set after an isotopy. The dividing set $\Gamma_{\partial M_1}$ then consists of three parallel curves isotopic to the core curve of A^+ labeled 1, 2, 3 and three parallel curves isotopic to the core of A^- labeled 4, 5, 6.

We now make the next cut in the convex decomposition along $\delta \times I$, where $\delta \subset \Sigma \setminus \gamma$ is a properly embedded oriented arc which connects the two boundary components of $\Sigma \setminus \gamma$ (from A^+ to A^-). (Some rounding will have taken place, but we assume that has already been taken care of.) $\delta \times I$ will be given the orientation induced from δ and I. We now consider the dividing curve configurations on $\delta \times I$. $\partial(\delta \times I)$ will intersect $\Gamma_{\partial M_1}$ in three points along A^+ and in three points along A^-. We label these points of intersection using the labels on the dividing curves. We claim that if there exists a ∂-parallel dividing curve straddling one of Positions 2, 3, 4, or 5, then the bypass corresponding to any of these positions, when considered back on A, would give a bypass along A (from one of the sides) and a new convex annulus isotopic to A with fewer dividing curves. Positions 2 and 5 give rise to bypasses whose Legendrian arcs of attachment are contained in A and which intersect three distinct curves of Γ_A. A bypass at Positions 3 or 4, when traced back to Figure 8(A), also yields a bypass along A which reduces $\#\Gamma_A$. To realize this, we apply Bypass Sliding (Lemma 2.3). Now, Figure 8(D) is the only remaining dividing curve configuration for $\delta \times I$ in which, potentially, there is no reduction to case II_0^+.

We proceed by rounding the edges of $M_1 \setminus (\delta \times I)$ to get M_2, depicted in Figure 8(E). This, after the unique dividing curve is straightened, is equivalent to Figure 8(F). In other words, $\Gamma_{\partial M_2}$ consists of one curve, and it separates ∂M_2.

Using (the proof of) Lemma 2.4 with $F = \partial M_2$, we can show the existence of a long bypass from the interior of ∂M_2 along the dotted line as shown in Figure 8(F). This implies the existence of a bypass in the interior of M_2 attached along the dotted line shown in Figure 8(E). This bypass can be used to isotop $\delta \times I$ to obtain a new configuration with bypasses in Positions 3 and 4. Therefore, we can always reduce from $II_{2m}^{\pm}$ to $II_0^{\pm}$. q.e.d.

Lemma 5.5. $\Gamma_A = II_0^{\pm}$ *extends uniquely to a tight contact structure on M. It is universally tight.*

Proof. After cutting M along A, we obtain $M \setminus A = S \times I$, where S is a surface of genus $g-1$ with two punctures. Applying edge-rounding, we obtain that $\Gamma_{\partial(S\times I)}$ is isotopic to $(\partial S) \times \{\frac{1}{2}\}$. Lemma 5.2 (or the proof of Lemma 5.2) implies that there is a unique tight contact structure which extends to the interior of $S \times I$. The tight contact structure is universally tight by Lemma 5.2, and glues to give a universally tight contact structure on M, since Γ_A is ∂-parallel and we can therefore apply Theorem 5.3. q.e.d.

5.2 Type $II_n^{\pm}$, n odd.

Lemma 5.6. *$II_{2m+1}^{\pm}$, $m \in \mathbb{Z}^+$, can be reduced to $II_1^{\pm}$.*

Proof. The proof is similar to the proof of Lemma 5.4. It is enough to show that $II_{2m+1}^{\pm}$ can be reduced to $II_{2m-1}^{\pm}$. Figure 9(A) depicts $M \setminus A$ where we have II_{2m+1}^{+}. After rounding the edges, we obtain M_1 which has $2m+3$ closed curves parallel to the core curve of A^+ and $2m+1$ closed curves parallel to the core curve of A^-. We take the next convex decomposing disk $\delta \times I$ with efficient Legendrian boundary. $\partial(\delta \times I)$ intersects $\Gamma_{\partial M_1}$ in $2m+3$ points along A^+, labeled 1 through $2m+3$ from closest to $\delta \times \{1\}$ to farthest from $\delta \times \{1\}$, and $2m+1$ points along A^-, labeled $2m+4$ through $4m+4$. The only ∂-parallel dividing curves on $\delta \times I$ which do not immediately lead to a bypass on A are those straddling Positions 1 and $2m+3$. Therefore, we are left to consider a unique choice for $\Gamma_{\delta \times I}$, given in Figure 9(C), i.e., exactly two ∂-parallel arcs (along 1 and $2m+3$), and all other dividing curves consecutively nested around them.

In order to prove the reduction from $II_{2m+1}^{\pm}$ to $II_{2m-1}^{\pm}$, we show the existence of a nontrivial bypass along $(\delta \times I)^-$ from the interior of M_2. Indeed, Lemma 2.4 guarantees that there is a bypass along an arc of attachment which intersects the ∂-parallel component of $(\delta \times I)^-$ straddling Position 1, as well as two consecutively nested dividing arcs. See Figure 9(D). This bypass, if viewed on $\delta \times I$ (Figure 9(C)), produces ∂-parallel curves across Positions $2m+2$ and $4m+3$, giving us a reduction in the number of parallel curves on Γ_A. q.e.d.

Lemma 5.7. *$II_1^{\pm}$ can be reduced to I_k.*

Proof. We will treat $\Gamma_A = II_1^-$, and leave II_1^+ to the reader. The computation is similar in spirit to the previous computations, except that it is a bit more involved. The goal is to find a bypass along A^+ from the interior of M_1 which straddles the three components of Γ_{A^+}. This time, the holonomy of the bypass (how many times the bypass wraps around the core curve) is important. In order to determine the existence of a bypass, we will successively cut M_1, leaving A^+ untouched, until we arrive at a solid torus whose boundary contains A^+. On the

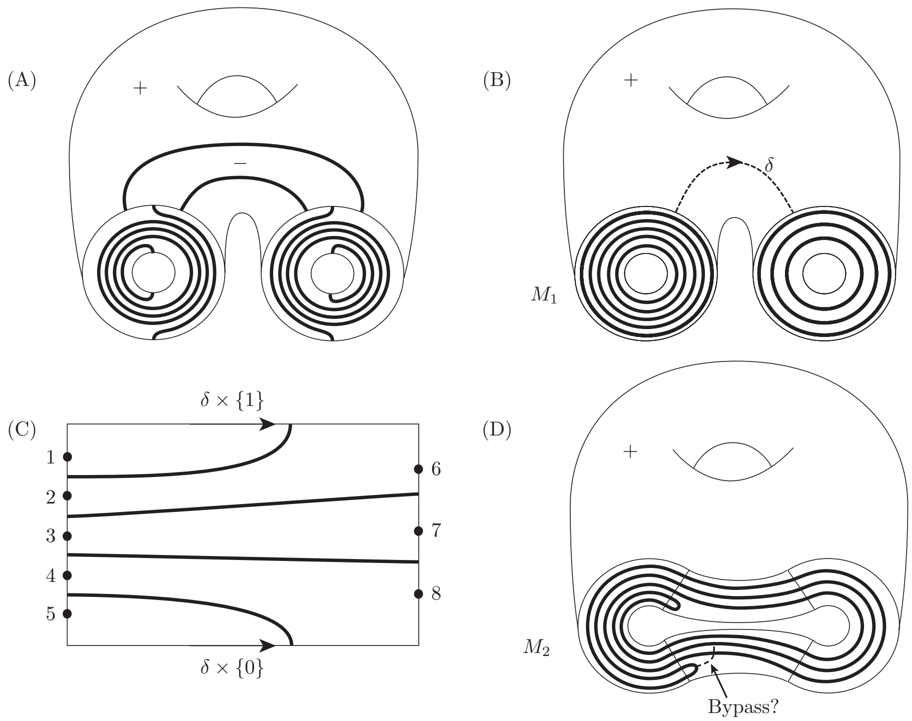

Figure 9: The case II^+_{2m+1}.

solid torus, we can determine whether the bypass exists, by appealing to the classification of tight contact structures on solid tori [11, 14].

Figures 10 and 11 depict this decomposition process. As shown in Figure 10(B), let δ_1 be a non-boundary-parallel, properly embedded arc on $\Sigma \backslash \gamma$ which does not intersect δ and begins and ends on ∂A^-. We take $\partial(\delta_1 \times I)$ to be Legendrian and efficient with respect to $\Gamma_{\partial M_1 \backslash A^+}$ on $\partial M_1 \backslash A^+$. Note this happens to be the same as being efficient with respect to $\Gamma_{\partial M_1}$ on ∂M_1. The side of Figure 10(A) which is hidden, namely $(\Sigma \setminus \gamma) \times \{0\}$, is as shown in Figure 8(B), that is, the hidden side differs from the top by a single Dehn twist. However, for subsequent figures, we suppose that the ∂-parallel dividing curve on A^- along $(\gamma \times \{0\})^-$ has already been twisted around the core curve of A^-, i.e., $\Gamma_{(\Sigma \backslash \gamma) \times \{0\}}$ is the same as $\Gamma_{(\Sigma \backslash \gamma) \times \{1\}}$. Now, take $\delta_1 \times I$ to be convex and define M_2 to be $M_1 \setminus (\delta_1 \times I)$, after rounding the edges. (Warning: This M_2 is different from the M_2's in the previous lemmas.) We label $\partial(\delta_1 \times I) \cap \Gamma_{\partial M_1}$ by numbers 1 through 6 as follows: There is a unique closed

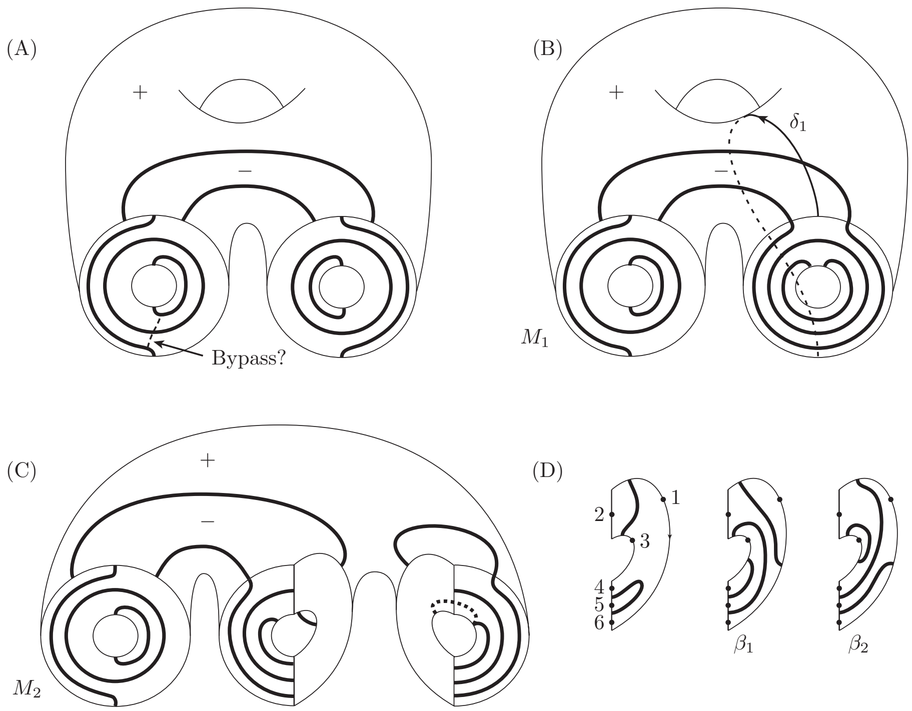

Figure 10: The case II_1^-.

curve of $\Gamma_{\partial M_1 \setminus A^+}$, and we let the two points of intersection of this curve with $\partial(\delta_1 \times I)$ be 2 and 5. The rest are labeled in increasing order along $\partial(\delta_1 \times I)$ in the direction given by the induced orientation, i.e., "counterclockwise". Now, ∂-parallel dividing curves in Positions 2 and 5 would immediately allow us to reduce to I_k, as can be seen on the $(\delta_1 \times I)^-$-side. Therefore, we are left with two possibilities for $\Gamma_{\delta_1 \times I}$, which we call β_1 and β_2. In Figure 10(D), the left diagram represents the two ∂-parallel positions which are ruled out, and the middle and right respectively are β_1 and β_2.

Consider β_1. (See Figure 11.) Figure 11(A) represents the dividing set $\Gamma_{\partial M_2}$ before rounding, and Figure 11(B) is $\Gamma_{\partial M_2}$ after rounding. We take δ_2 (as in Figure 11(B)) to be a non-boundary-parallel, properly embedded arc on $\Sigma \setminus (\gamma \cup \delta_1)$ which begins on one copy of δ_1 and ends on the other copy. At this point, $\Sigma \setminus (\gamma \cup \delta_1)$ is a pair-of-pants, and cutting along δ_2 yields an annulus. Take $\partial(\delta_2 \times I)$ to be Legendrian and efficient with respect to $\Gamma_{\partial M_2 \setminus A^+}$ (which also happens to be the same as being efficient with respect to $\Gamma_{\partial M_2}$). Now, $tb(\partial(\delta_2 \times I)) = -2$,

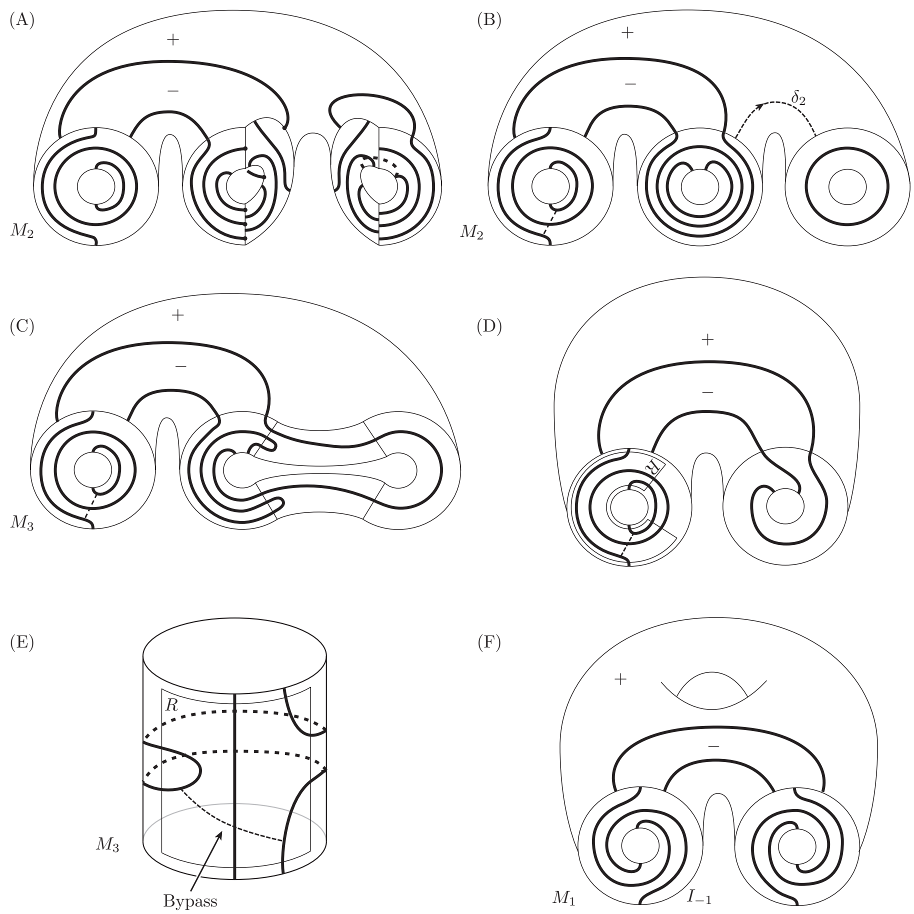

Figure 11: The case II_1^-, continued. Configuration β_1.

and there are two possibilities for $\Gamma_{\delta_2 \times I}$. We may rule out one of the possibilities, since it yields a ∂-parallel dividing curve which allows us to reduce to I_k.

Finally, we have a solid torus $M_3 = M_2 \setminus (\delta_2 \times I)$, and $\Gamma_{\partial M_3}$ consists of 2 parallel longitudinal dividing curves. (Figures 11(C,D,E) represent $\Gamma_{\partial M_3}$ before and after successive edge-roundings.) A compressing disk D intersecting each dividing curve once may be chosen so that $M_3 \setminus D$ is as shown in Figure 11(E), and in particular so that the rectangles labeled R in Figure 11(D) and Figure 11(E) correspond. This implies, by [14], that M_3 is a standard neighborhood of a Legendrian curve. Let us identify $\partial M_3 = \mathbb{R}^2/\mathbb{Z}^2$ by letting the meridian have slope 0 and $\Gamma_{\partial M_3}$

have slope ∞.

We claim that there exists a bypass along A^+ from the interior of M_1, which changes II_1^- to I_{-1}. We look for the corresponding bypass along A^+ on M_3. To see this exists, let D be a convex meridional disk for M_3 with Legendrian boundary ∂D which is efficient with respect to $\Gamma_{\partial M_3}$ and is disjoint from the bypass arc of attachment. Then $tb(\partial D) = -1$, and there is a unique way, up to isotopy, to cut this solid torus into a 3-ball B^3. Finally, the bypass along ∂B^3 is a trivial bypass, which must therefore exist by Right-to-Life. q.e.d.

5.3 Type I_k.

Lemma 5.8. *I_k, $k > 0$, can be reduced to I_0.*

Proof. The decomposition procedure is given in Figure 12. Figure 12(A) gives $M \setminus A$; note that in this figure $\Gamma_{(\Sigma\setminus\gamma)\times\{0\}}$ does not equal $\Gamma_{(\Sigma\setminus\gamma)\times\{1\}}$ and rather is as in Figure 8(B). Figure 12(B) is the same as Figure 12(A), except that the extra Dehn twist is incorporated in A^- so that $\Gamma_{(\Sigma\setminus\gamma)\times\{0\}} = \Gamma_{(\Sigma\setminus\gamma)\times\{1\}}$. Figure 12(C) depicts ∂M_1, where M_1 is $M \setminus A$ with the edges rounded.

Let $\delta \subset \Sigma \setminus \gamma$ be the same as in Lemma 5.4. As before, consider the compressing disk $\delta \times I$, which we take to be convex with efficient Legendrian boundary. $\delta \times I$ is chosen so that $\partial(\delta \times I)$ intersects $\Gamma_{\partial M_1}$ in $2k-1$ points along A^+ (labeled 1 through $2k-1$ in order from closest to $\delta \times \{1\}$ to farthest) and $2k+3$ points along A^- (labeled $2k$ through $4k+2$ in order from closest to $\delta \times \{1\}$ to farthest). If there are ∂-parallel components of $\Gamma_{\delta\times I}$ along any of $2k+1, \ldots, 4k+1$, then the corresponding bypasses would give rise to the state transition from I_k to I_{k-1}. Now, there are $2k+2$ endpoints of $\Gamma_{\delta\times I} \cap \partial(\delta\times I)$ between Positions $2k$ and $4k+2$. If there are connections (dividing arcs) amongst the $2k+2$ endpoints, then clearly, this would give rise to a ∂-parallel arc straddling one of the "reducing" positions. However, this must happen since the total number of endpoints is $4k+2$, i.e., $\#(\Gamma_{\delta\times I} \cap \partial(\delta \times I)) = 4k+2$, and $4k+2 < 2(2k+2)$. q.e.d.

Lemma 5.9. *I_k, $k < -1$, can be reduced to I_{-1}.*

The apparent lack of symmetry between Lemmas 5.8 and 5.9 is due to the fact that the first cut along A is not symmetric with respect to Γ_0.

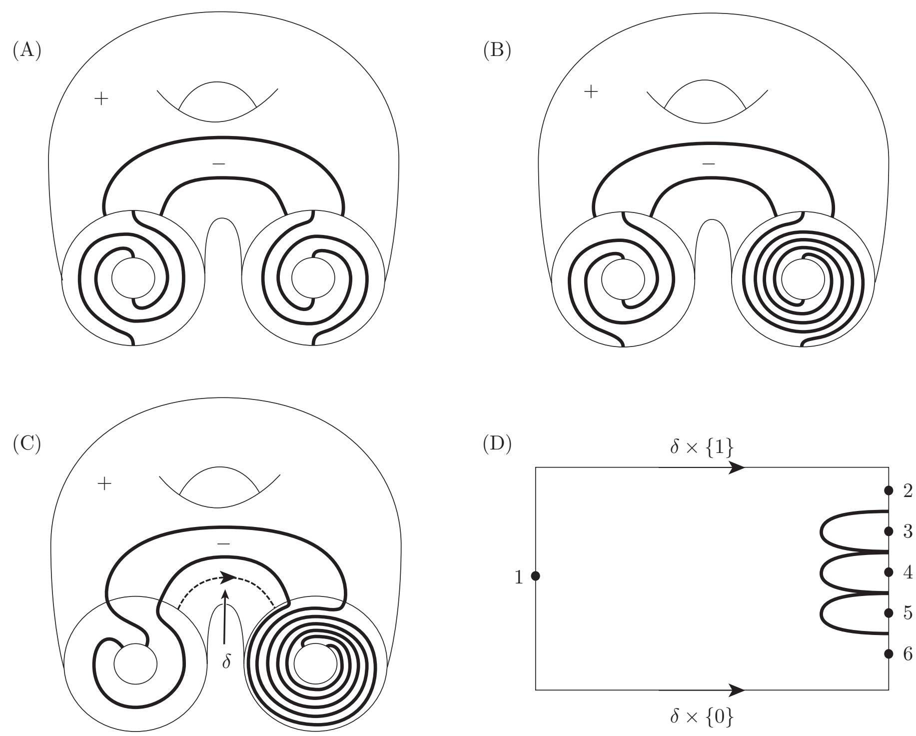

Figure 12: The case I_k, $k > 0$.

Proof. The argument is almost identical to that of Lemma 5.8. The only difference is that the computations are mirror images of those of Lemma 5.8. Refer to Figure 13 for the steps of the computation. q.e.d.

Lemmas 5.8 and 5.9 together indicate that all the I_k's can be reduced to either I_0 or I_{-1}. The following Lemma computes (an upper bound for) the tight contact structures where $\Gamma_A = I_0$, and relates them to I_{-1}.

Lemma 5.10. *There are at most two tight contact structures on $\Sigma \times I$ in the Base Case for which $\Gamma_A = I_0$. The same also holds for $\Gamma_A = I_{-1}$. There exist state transitions along A which allow us to switch between I_0 and I_{-1}.*

Proof. Let us take the case $\Gamma_A = I_0$. Figure 14(A) gives the dividing set of $M \setminus A$, before edge-rounding but after the extra Dehn twist from the bottom face $(\Sigma \setminus \gamma) \times \{0\}$ is included. (Therefore, $\Gamma_{(\Sigma\setminus\gamma)\times\{0\}} = \Gamma_{(\Sigma\setminus\gamma)\times\{1\}}$.) Figure 14(B) is the same after edge-rounding. Now, if we

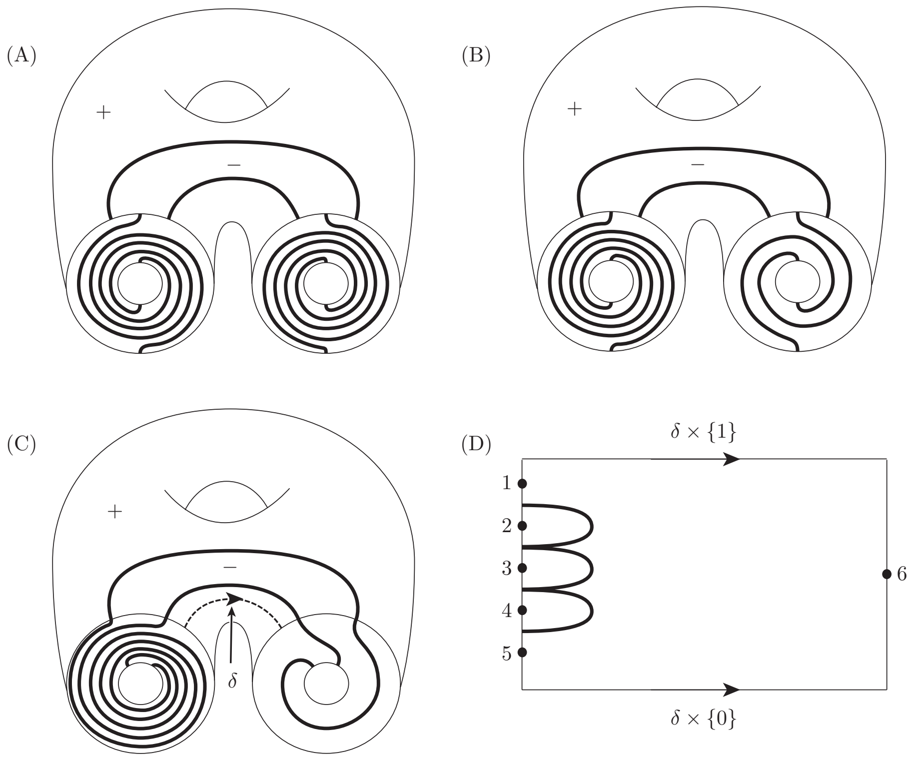

Figure 13: The case I_k, $k < -1$.

cut along $\delta \times I$, defined as in Lemma 5.8, there are two possibilities for $\Gamma_{\delta\times I}$, since $tb(\partial(\delta \times I)) = -2$. (These are shown in Figures 14(C,D).) Figures 14(E,F) depict the dividing set of $M_2 = M_1 \setminus (\delta \times I)$, after edge-rounding. In both cases, $\Gamma_{\partial M_2}$ consists of exactly one dividing curve parallel to $\partial(\Sigma \setminus (\gamma \cup \delta))$. Finally, using Lemma 5.2, we find that for each of the two possibilities of $\Gamma_{\delta\times I}$ there is a unique universally tight contact structure on M_2. Theorem 5.3 is also sufficient to glue back along $\delta \times I$ to give two universally tight contact structures on M_1 with the boundary condition given by Figure 14(A). Making the final gluing and proving the resulting contact structure is universally tight is a more complicated state-transition operation, so we will content ourselves for the time being with the knowledge that there are at most two tight contact structures on $\Sigma \times I$ in the Base Case with $\Gamma_A = I_0$. A similar computation also holds for $\Gamma_A = I_{-1}$.

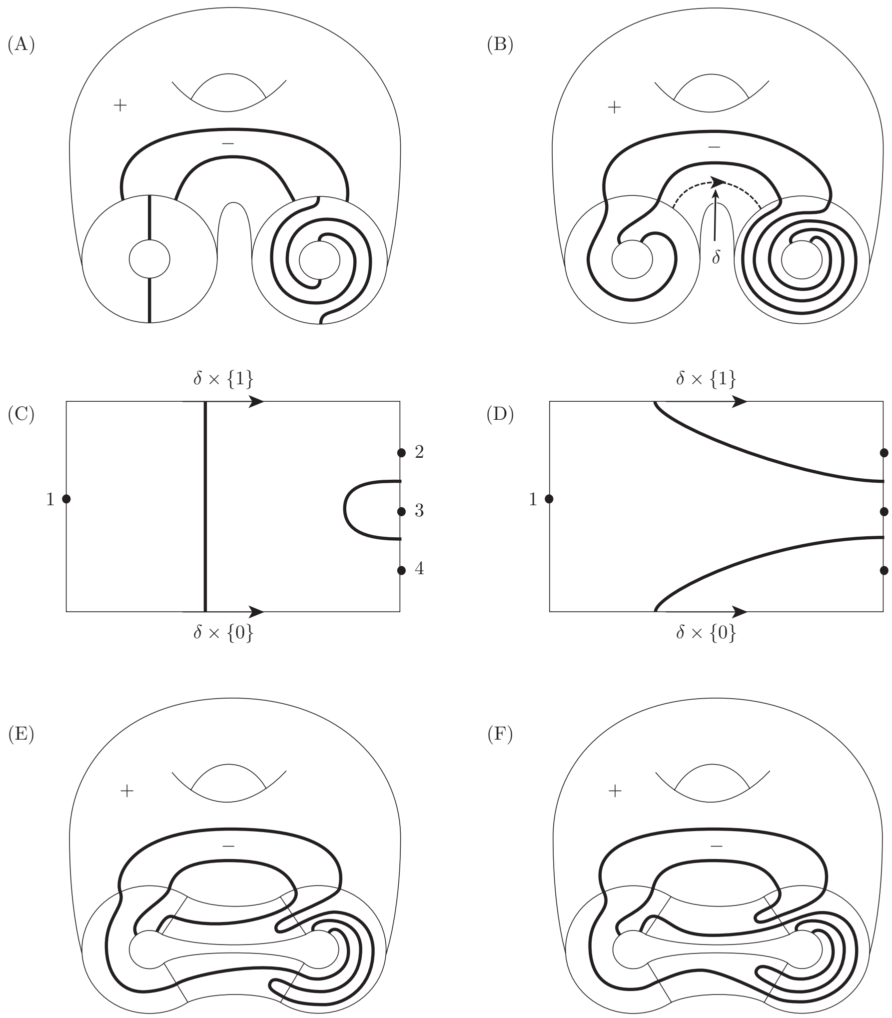

Figure 14: Classification for I_0.

We now prove that we may switch from I_0 to I_{-1}. The reverse procedure is identical. Above, we found that $\#(\Gamma_{\partial M_1} \cap \partial(\delta \times I)) = 4$, and one intersection was on A^+ (labeled 1) and 3 on A^- (labeled $2, 3, 4$ in succession from closest to $\delta \times \{1\}$ to farthest). A ∂-parallel dividing curve of $\delta \times I$ straddling Position 3 clearly allows us to transition from I_0 to I_{-1}. On the other hand, a ∂-parallel dividing curve straddling Position 4 gives rise to a corresponding bypass which can be slid so that all three of the intersections with $\Gamma_{\partial M_1}$ lie on A^-. Therefore, for both choices of $\Gamma_{\delta\times I}$, we may transition from I_0 to I_{-1}. q.e.d.

5.4 Completion of Theorem 5.1.

Summarizing what we have proved so far:

- $\#\pi_0(\text{Tight}\,(M,\mathcal{F})) \leq 4$.
- Type II_0^+: there exists one.
- Type II_0^-: there exists one.
- Type I_0: there are at most 2.
- The other possibilities for Γ_A reduce to one of $II_0^{\pm}$ or I_0.
- The tight contact structures of Type $II_0^{\pm}$ are universally tight by Lemma 5.5.
- The tight contact structures of Type II_0^+ and II_0^- are distinct and are also distinct from those of Type I_0 by the relative Euler class $\widetilde{e}$ evaluated on A.

The proof of Theorem 5.1 is complete, once we show Lemmas 5.11 and 5.13 below.

Lemma 5.11. *There are exactly two tight contact structures of Type I_0. They are universally tight, are obtained by adding a single bypass onto Σ_0, and have relative Euler class* $\mathrm{PD}(\widetilde{e}(\xi)) = \pm(\gamma_1 - \gamma_0)$.

Proof. We show the existence of the (universally) tight contact structures by embedding them inside a suitable (universally) tight contact structure of Type $II_0^{\pm}$. Let $A = \gamma \times I$ be the first cut that was used to decompose $\Sigma \times I$. If $\Gamma_A = II_0^+$, then there is a ∂-parallel dividing curve along $\gamma \times \{0\}$ which cuts off a positive region of A. There is a corresponding degenerate bypass whose Legendrian arc of attachment is all of $\gamma \times \{0\}$. (Degenerate means that the two ends of the Legendrian arc of attachment are identical [14].) If we immediately attach this degenerate bypass onto Σ_0, we obtain an isotopic convex surface which we call $\Sigma'_{1/2}$ and which satisfies $\Gamma_{\Sigma'_{1/2}} = 2\gamma$. However, if we separate the endpoints of the degenerate arc of attachment by bypass sliding in one particular direction along Γ_{Σ_0}, we obtain a more convenient convex surface $\Sigma_{1/2}$ with $\Gamma_{\Sigma_{1/2}} = 2\gamma_1$, after the bypass attachment. The contact structure on $\Sigma \times [0, 1/2]$ is universally tight since it is a subset of the universally tight contact structure of Type II_0^+. Starting from the tight contact structure of Type II_0^- and using the same argument, we can produce another universally tight contact structure.

We now compute the relative Euler classes of the universally tight contact structures constructed in the previous paragraph. It will then be clear that the two tight contact structures are distinct and of Type I_0. We fix some notation. Let ξ be a tight contact structure obtained in the previous paragraph with ambient manifold $\Sigma \times [0, 1]$ and $\Gamma_{\Sigma_i} = 2\gamma_i$, $i = 0, 1$. Also suppose that ξ is $[0, 1]$-invariant except for $\mathcal{A} \times [0, 1]$, where $\mathcal{A} \subset \Sigma_0$ is a convex annulus with Legendrian boundary whose core curve is isotopic to γ, and the bypass was attached to Σ_0 along an arc in $\mathcal{A}$.

First suppose $\beta \subset \Sigma$ is a closed nonseparating curve which intersects neither γ_0 nor γ_1. Then the corresponding convex annulus $\beta \times I$ will only consist of closed curves parallel to the core curve. Hence $\langle \widetilde{e}(\xi), \beta \times I \rangle = 0$. Thus we may choose a basis of $H_1(\Sigma; \mathbb{Z})$ such that, of the $2g$ generators, $2g - 2$ of them evaluate to zero in this manner. Next let $\beta \times \{0\} \subset \Sigma_0$ be a closed Legendrian curve parallel to γ but disjoint from $\mathcal{A}$. Then, since ξ is I-invariant away from $\mathcal{A} \times I$, $\beta \times I$ must consist of 2 parallel vertical nonseparating arcs, that is, $\langle \widetilde{e}(\xi), \beta \times I \rangle = 0$ and this shows that ξ on $\Sigma \times [0, 1/2]$ is of Type I_0. Finally, let $\beta \times \{0\} \subset \Sigma_0$ be a closed efficient Legendrian curve which is parallel to γ_1 but does not intersect the arc of attachment of the bypass, which we take to be nondegenerate. Then $\beta \times \{i\}$, $i = 0, 1$, intersects Γ_{Σ_i} twice, and $\Gamma_{\beta \times I}$ consists of 2 parallel vertical nonseparating arcs. However, now $\beta \times \{1\}$ is not efficient with respect to Γ_{Σ_i}, and resolving the extra intersection to produce an efficient intersection gives $\langle \widetilde{e}(\xi), (\beta \times I)^* \rangle = \pm 1$. (Here, $(\cdot)^*$ refers to the annulus with efficient Legendrian boundary.) See Figure 15. Having evaluated $\widetilde{e}(\xi)$ on all the basis elements, we find that $\mathrm{PD}(\widetilde{e}(\xi)) = \pm\gamma = \pm(\gamma_1 - \gamma_0)$. q.e.d.

Definition 5.12. A tight contact structure on $\Sigma \times I$ which is contact diffeomorphic to one of the tight contact structures of Type I_0 is said to be a *basic slice*. Thus, in a basic slice, $\Gamma_{\Sigma \times \{0\}}$ and $\Gamma_{\Sigma \times \{1\}}$ each consist of two parallel curves $2\gamma_0$ and $2\gamma_1$ with $|\gamma_0 \cap \gamma_1| = 1$. The basic slice is obtained by attaching a single bypass B onto $\Sigma \times \{0\}$ and thickening $(\Sigma \times \{0\}) \cup B$.

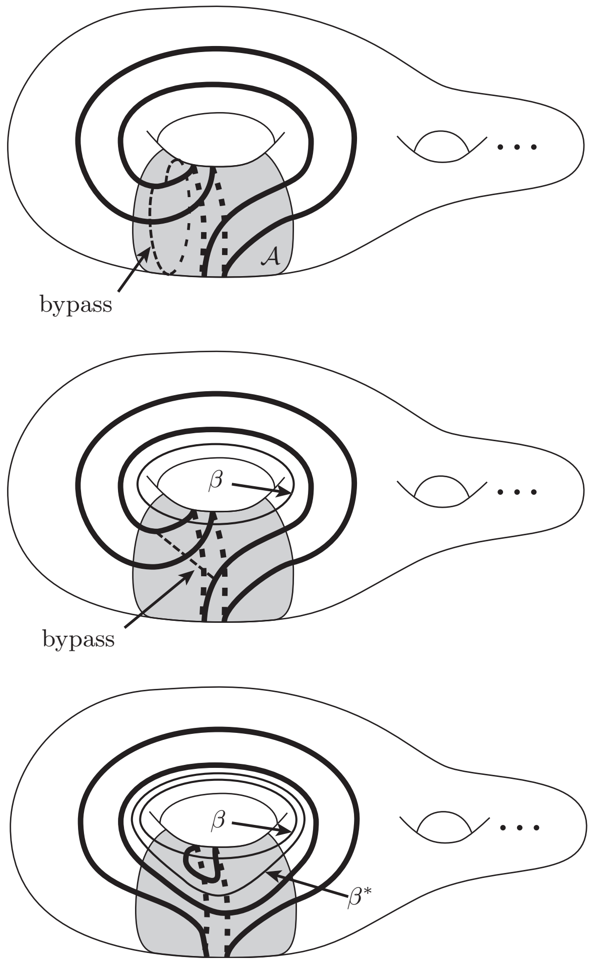

Figure 15: Ocarinian Riemann surfaces.

Lemma 5.13. *The tight contact structure of Type $II_0^{\pm}$ has relative Euler class* $\mathrm{PD}(\widetilde{e}(\xi)) = \mp(\gamma_1 + \gamma_0)$.

Proof. We will restrict our attention to II_0^+. As in the proof of Lemma 5.11, if $\beta \subset \Sigma$ is a closed nonseparating curve which does not intersect γ_0 or γ_1, then $\langle \widetilde{e}(\xi), \beta \times I \rangle = 0$. From the definition of II_0^+ we have:

$$\langle \widetilde{e}(\xi), (\gamma_1 - \gamma_0) \times I \rangle = 2.$$

Next, since $\gamma_0 \times I$ with efficient Legendrian boundary intersects Γ_{Σ_0} zero times and Γ_{Σ_1} twice,

$$\langle \widetilde{e}(\xi), \gamma_0 \times I \rangle = \pm 1,$$

where the exact sign will be determined in a moment. Similarly,

$$\langle \widetilde{e}(\xi), \gamma_1 \times I \rangle = \pm 1.$$

For the three equations to agree, we must have $\langle \widetilde{e}(\xi), \gamma_0 \times I \rangle = -1$ and $\langle \widetilde{e}(\xi), \gamma_1 \times I \rangle = 1$. This implies that $\mathrm{PD}(\widetilde{e}(\xi)) = -(\gamma_1 + \gamma_0)$. q.e.d.

6. Classification of tight contact structures on $\boldsymbol{\Sigma \times I}$

In this section we prove Theorem 1.1 as well as the following theorem:

Theorem 6.1 (Gluing Theorem). *Let Σ be an oriented closed surface of genus $g \geq 2$, $M = \Sigma \times [0,2]$, and ξ a contact structure which is tight on $M \setminus \Sigma_1$. Suppose Σ_i, $i = 0, 1, 2$, are convex and $\Gamma_{\Sigma_i} = 2\gamma_i$, where γ_i are nonseparating oriented curves. Also assume that the γ_i are not mutually homologous. If $\mathrm{PD}(\widetilde{e}(\xi|_{\Sigma \times [0,1]})) = \gamma_1 - \gamma_0$ (here γ_0 and γ_1 have been oriented so the relative Euler class has this form), then $\mathrm{PD}(\widetilde{e}(\xi|_{\Sigma \times [1,2]})) = \gamma_2 - \gamma_1$ (for some orientation of γ_2) if and only if ξ is tight on M.*

The condition of the γ_i being mutually nonhomologous is a technical condition, which can be removed if we reformulate the Gluing Theorem without reference to the relative Euler class. The reader is encouraged to do so, after examining the proof of Theorem 1.1 and the Gluing Theorem. As we will see, the only contact topology calculations needed to prove Theorem 1.1 are the one done in Section 5 and a similar calculation in Proposition 6.4. The rest is largely a "proof by pure thought", relying on the relative Euler class consistency check, Proposition 6.2, and curve complex facts.

Notation. We will write $[\alpha_0, \alpha_1; \mathrm{PD}(\widetilde{e}(\xi))]$ to mean some tight contact structure on $\Sigma \times [0,1]$ with $\Gamma_{\Sigma_i} = 2\alpha_i$, $i = 0, 1$, and relative Euler class $\mathrm{PD}(\widetilde{e}(\xi))$. If we do not specify the relative Euler class (or it is understood), we simply write $[\alpha_0, \alpha_1]$. To indicate a basic slice, we write $[\![\alpha_0, \alpha_1]\!]$. We will use the notation $[\alpha_0, \alpha_1] = [\alpha_0, \alpha] \cup [\alpha, \alpha_1]$ to indicate that $\Sigma \times [0,1]$ has been split along a convex surface with dividing set 2α.

6.1 Freedom of choice

The proof of Theorem 1.1 is founded on the following rather remarkable proposition.

Proposition 6.2 (Freedom of choice). *Let $(M = \Sigma \times I, \xi)$ be a contact structure $[\gamma_0, \gamma_1]$ with $\langle \gamma_1, \gamma_0 \rangle = +1$, where $\langle \cdot \rangle$ represents the intersection form on Σ, a surface of genus at least 2. Let γ be any nonseparating curve on Σ. Then there exists a convex surface isotopic to Σ_0 in the interior of M which induces a splitting $[\gamma_0, \gamma_1] = [\gamma_0, \gamma] \cup [\gamma, \gamma_1]$.*

The strategy of the proof is to start with $\Gamma_{\Sigma_0} = 2\gamma_0$ and $\Gamma_{\Sigma_1} = 2\gamma_1$ and successively find subslices $\Sigma \times [a, b] \subset \Sigma \times [0, 1]$ with convex boundary and dividing sets Γ_{Σ_a}, Γ_{Σ_b} which consist of two parallel curves each and represent curves which are "closer" to γ inside the curve complex. To prevent our notation from becoming too cumbersome, we will rename the old $\Sigma \times [a, b]$ to be the new $\Sigma \times [0, 1]$ after each step of the induction.

The next lemma describes the splitting operation which will be used repeatedly in the proof.

Lemma 6.3. *Consider $[\alpha_0, \alpha_1]$, where $|\alpha_0 \cap \alpha_1| = 1$. Let α be a closed (necessarily nonseparating) curve which satisfies $|\alpha_0 \cap \alpha| = 0$ and $|\alpha_1 \cap \alpha| = 1$. Then there exists a splitting $[\alpha_0, \alpha_1] = [\alpha_0, \alpha] \cup [\![\alpha, \alpha_1]\!]$. Similarly if β is a closed curve such that $|\alpha_0 \cap \beta| = 1$ and $|\alpha_1 \cap \beta| = 0$, then there exists a splitting $[\alpha_0, \alpha_1] = [\![\alpha_0, \beta]\!] \cup [\beta, \alpha_1]$.*

Proof of Lemma. On Σ_0, use LeRP to realize $\alpha \times \{0\}$ as a Legendrian curve with $\#(\Gamma_{\Sigma_0} \cap \alpha) = 0$. Similarly, Legendrian realize $\alpha \times \{1\}$ with $\#(\Gamma_{\Sigma_1} \cap \alpha) = 2$. Take the convex annulus $\alpha \times I$. By the Imbalance Principle of [14], there must be a ∂-parallel dividing curve along $\alpha \times \{1\}$ and hence a degenerate bypass. Attaching the degenerate bypass gives an isotopic convex surface with $\Gamma = 2\alpha$. This surface gives the desired splitting. An analogous proof works for β. q.e.d.

Proof of Proposition 6.2. We start with $[\gamma_0, \gamma_1]$. Using Proposition 3.5, we obtain a sequence:

$$\alpha_0 = \gamma_0, \alpha_1, \alpha_2, \ldots, \alpha_k = \gamma_1,$$

where α_i, $i = 0, \ldots, k$, are nonseparating, $|\alpha_{i-1} \cap \alpha_i| = 1$, $i = 1, \ldots, k$, and $|\alpha_{i-1} \cap \alpha_{i+1}| = 0$, $i = 1, \ldots, k-1$. Now, using Lemma 6.3, we successively find:

$$[\alpha_0, \alpha_1] \supset [\![\alpha_2, \alpha_1]\!] \supset [\![\alpha_2, \alpha_3]\!] \supset \cdots \supset [\![\alpha_{k-1}, \alpha_k]\!].$$

This completes the proof of the proposition. q.e.d.

6.2 Proof of Part (3) of Theorem 1.1.

We will now give the classification result for $[\gamma_0, \gamma_1 = \gamma_0]$, where γ_0 is an oriented nonseparating curve. If the tight contact structure ξ is I-invariant, then $\mathrm{PD}(\widetilde{e}(\xi)) = 0$.

Proposition 6.4. *Let $[\gamma_0, \gamma_0]$ be a tight contact structure on $M = \Sigma \times [0,1]$ which is not I-invariant. If γ is an oriented nonseparating curve on Σ which satisfies $\langle \gamma, \gamma_0 \rangle = +1$, then there exist splitting of $[\gamma_0, \gamma_0]$ as $[\![\gamma_0, \gamma]\!] \cup [\gamma, \gamma_0]$. Alternatively, the splitting can be chosen to be $[\gamma_0, \gamma] \cup [\![\gamma, \gamma_0]\!]$.*

Proof. The proof is a calculation along the lines of Section 5. Consider the convex annulus $A = \gamma \times I$ with efficient Legendrian boundary. Γ_A has the possibilities as in Figure 7, denoted types I_k and $II_n^{\pm}$. Types $II_n^{\pm}$ all have ∂-parallel dividing curves which give rise to bypasses and which, in turn, yield basic slices. Therefore, it suffices to consider I_k.

If $\Gamma_A = I_0$, then there exists a sequence of convex meridional disks D_i which decompose $M_1 = M \setminus A$ into the 3-ball, and for which $tb(\partial D_i) = -1$. Hence, there must be a unique tight contact structure with $\Gamma_A = I_0$. This implies that $\Gamma_A = I_0$ represents the I-invariant case.

Suppose $\Gamma_A = I_k$ with $k > 0$. Let $M_1 = M \setminus A$. Figure 16(A) gives M_1 and Figure 16(B) the same with the edges of A^- rounded. (Note that Figure 16 depicts the case $\Gamma_A = I_1$.) They are presented in almost identical fashion as in Section 5 with one exception: now $\Gamma_{(\Sigma\setminus\gamma)\times\{1\}} = \Gamma_{(\Sigma\setminus\gamma)\times\{0\}}$. Our notation will be identical to that of Lemma 5.7. Let δ_1 be a non-boundary-parallel, properly embedded arc in $\Sigma \setminus \gamma$ which does not intersect Γ_Σ and satisfies $\partial\delta_1 = p_1 - p_0$, where $p_0, p_1 \in \partial A^-$. Take $\delta_1 \times I$ and perturb it to be convex with efficient Legendrian boundary so that $|\partial(\delta_1 \times I) \cap \Gamma_{\partial M_1}| = 4k + 2$, and $2k + 1$ of the intersections we may assume are on $p_0 \times I$ (labeled $1, \dots, 2k + 1$ from closest to $\delta_1 \times \{1\}$ to farthest) and the other $2k + 1$ on $p_1 \times I$ (labeled $2k + 2, \dots, 4k+2$ from farthest from $\delta_1 \times \{1\}$ to closest). If there is a ∂-parallel dividing arc on $\delta_1 \times I$ straddling Positions $2, \dots, 2k$ or $2k+3, \dots, 4k+1$, then the corresponding bypass state transitions us into I_{k-1}. If we can continue this, we eventually get to I_0, which is already taken care of. (Actually, it is unlikely such a state transition exists; we probably have an overtwisted contact structure here.) Otherwise, we have two possibilities: β_1 and β_2.

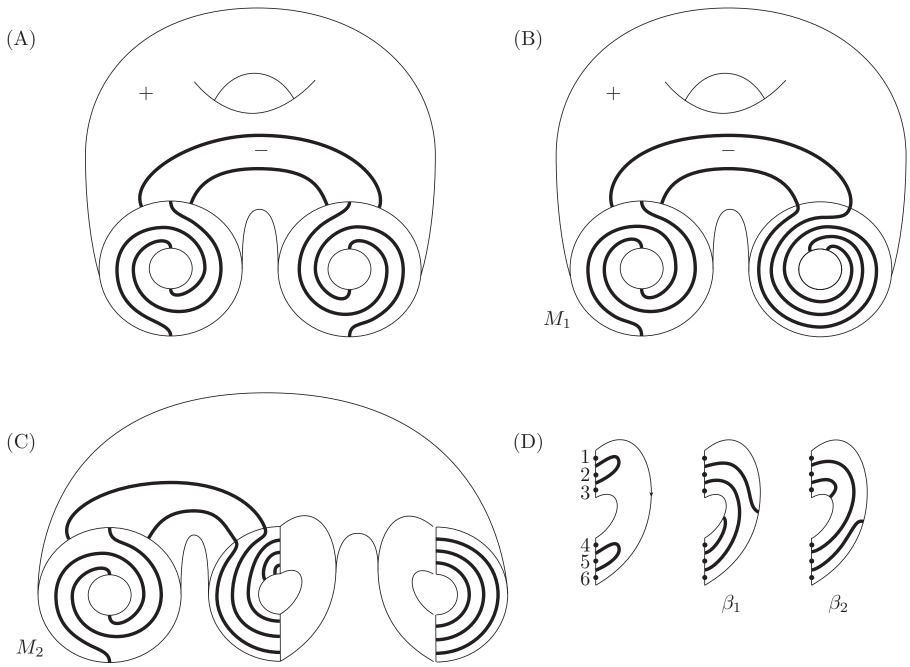

Figure 16: Transition from I_k to $II_n^{\pm}$.

The case of β_1 occupies Figures 17(A,B) and the case of β_2 occupies Figure 17(C). We will explain β_2 first. After rounding the edges, $\Gamma_{\partial M_2 \setminus A^+}$ has one ∂-parallel arc along each boundary component of A^+. This implies that there exists a Legendrian divide α_1 on $\partial M_2 \setminus A^+$ parallel to either boundary component of A^+ (after possibly perturbing $\partial M_2 \setminus A^+$ as in LeRP). Let α_2 be an efficient Legendrian curve on A^+, isotopic to the core curve and satisfying $|\alpha_2 \cap \Gamma_{A^+}| = 2$. If we take an annulus spanning α_1 to α_2, by the Imbalance Principle we will obtain a degenerate bypass along α_2. Attaching the degenerate bypass will give the transition to some $II_n^{\pm}$.

On the other hand, β_1 does not immediately give rise to a ∂-parallel arc along ∂A^+. Therefore, we cut again, this time along $\delta_2 \times I$, which is convex with efficient Legendrian boundary, to obtain M_3. Write $\partial \delta_2 = q_1 - q_0$. Now, $|\partial(\delta_2 \times I) \cap \Gamma_{\partial M_2}| = 2k+2$, and $2k+1$ of the intersections are on $q_0 \times I$, whereas 1 intersection is on $q_1 \times I$. If $k > 1$, then there will always be a bypass along $q_0 \times I$, which transitions us to I_{k-1}.

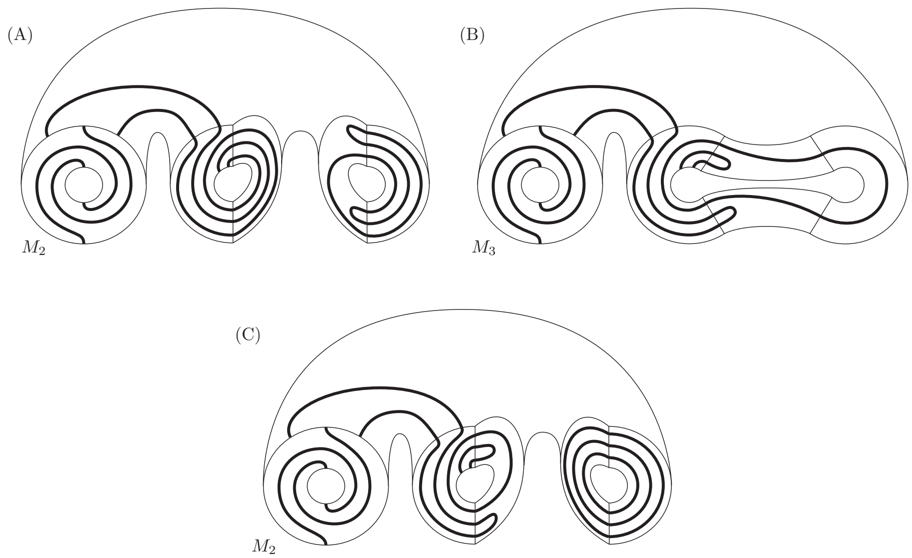

Figure 17: Transition from I_k to $II_n^{\pm}$, continued.

On the other hand, there is one extra case when $k = 1$ — after the edges are rounded (Figure 17(B)), there exists a ∂-parallel arc along ∂A^+ on $\partial M_3 \setminus A^+$. Therefore, we will always have a state transition to some $II_n^{\pm}$, provided I_k does not represent an I-invariant tight contact structure. q.e.d.

We claim that $\mathrm{PD}(\widetilde{e}([\gamma_0, \gamma_0])) = \pm 2\gamma_0$ or 0. To see this, first note that if β is any curve with $|\gamma_0 \cap \beta| = 0$, then $\langle \widetilde{e}([\gamma_0, \gamma_0]), \beta \times I\rangle = 0$, since $\Gamma_{\beta \times I}$ consists solely of closed curves parallel to the core curve. This takes care of $2g - 1$ generators of $H_1(\Sigma; \mathbb{Z})$. Next, if β satisfies $|\gamma_0 \cap \beta| = 1$, then $\langle \widetilde{e}([\gamma_0, \gamma_0]), \beta \times I\rangle$ is -2, 0, or 2, depending on the configuration of ∂-parallel dividing curves on $\beta \times I$. This proves the claim.

Suppose now that $[\gamma_0, \gamma_0]$ is not I-invariant. Let $\gamma \subset \Sigma$ be a closed oriented curve satisfying $\langle \gamma_0, \gamma\rangle = +1$. By Proposition 6.4, there exists a factorization into $[\gamma_0, \gamma] \cup [\![\gamma, \gamma_0; \pm(\gamma_0 - \gamma)]\!]$. (Notation: when we write $[a_1, a_2] \cup [a_2, a_3] \cup \cdots \cup [a_{k-1}, a_k]$, the contact structure is layered in order.) We initially have the possibilities $\mathrm{PD}(\widetilde{e}([\gamma_0, \gamma])) = \pm\gamma_0 \pm \gamma$ from Theorem 5.1. However, by the claim in the previous paragraph, if $\mathrm{PD}(\widetilde{e}([\![\gamma, \gamma_0]\!])) = \gamma_0 - \gamma$, then $\mathrm{PD}(\widetilde{e}([\gamma_0, \gamma])) = \pm\gamma_0 + \gamma$. Therefore, there is a total of 4 possibilities:

$$[\gamma_0, \gamma_0] = [\gamma_0, \gamma; \pm\gamma_0 + \gamma] \cup [\![\gamma, \gamma_0; \gamma_0 - \gamma]\!],$$

or

$$[\gamma_0, \gamma_0] = [\gamma_0, \gamma; \pm\gamma_0 - \gamma] \cup [\![\gamma, \gamma_0; -(\gamma_0 - \gamma)]\!].$$

Lemma 6.5. *All four contact structures are tight, distinct, and can be embedded in a basic slice.*

Proof. Let γ_0, γ_1 be nonseparating curves satisfying $\langle \gamma_1, \gamma_0 \rangle = +1$. We claim that $[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!]$ can be factored into $[\gamma_0, \gamma_0] \cup [\![\gamma_0, \gamma_1]\!]$, where the first factor is *not* I-invariant. First we layer $[\![\gamma_0, \gamma_1]\!] = [\gamma_0, \alpha] \cup [\alpha, \gamma_1]$, where $|\alpha \cap \gamma_i| = 1$, $i = 0, 1$, using Proposition 6.2.

Next consider the layer $[\alpha, \gamma_1]$. Again, by Proposition 6.2, we may split so that $[\alpha, \gamma_1] = [\alpha, \beta] \cup [\beta, \gamma_1]$, where β is chosen so that $|\beta \cap \gamma_0| = 0$ and $|\beta \cap \gamma_1| = 1$. By Lemma 6.3 applied to $[\beta, \gamma_1]$, we can split so that $[\beta, \gamma_1] = [\beta, \gamma_0] \cup [\![\gamma_0, \gamma_1]\!]$.

This gives

$$\begin{aligned} [\![\gamma_0, \gamma_1]\!] &= [\gamma_0, \alpha] \cup [\alpha, \gamma_1] = [\gamma_0, \alpha] \cup [\alpha, \beta] \cup [\beta, \gamma_1] \\ &= [\gamma_0, \alpha] \cup [\alpha, \beta] \cup [\beta, \gamma_0] \cup [\![\gamma_0, \gamma_1]\!]. \end{aligned}$$

By combining the first three slices, we get a layer $[\gamma_0, \gamma_0]$ which cannot be I-invariant by the semi-local Thurston-Bennequin inequality (see [12]). We now have

$$[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!] = [\gamma_0, \gamma_0] \cup [\![\gamma_0, \gamma_1]\!].$$

It remains to compute $\mathrm{PD}(\widetilde{e})$ of the second factor. If we reconcile $\mathrm{PD} = \pm 2\gamma_0$ or 0 for the first factor with $\mathrm{PD} = \pm(\gamma_1 - \gamma_0)$ for the second factor, we easily see that:

$$[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!] = [\gamma_0, \gamma_0; 0] \cup [\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!].$$

We therefore have realized at least one tight contact structure $[\gamma_0, \gamma_0; 0]$ which is not I-invariant. We can also obtain another non-I-invariant tight contact structure $[\gamma_0, \gamma_0; 0]$ by starting from $[\![\gamma_0, \gamma_1; \gamma_0 - \gamma_1]\!]$.

Our next claim is that the two non-I-invariant $[\gamma_0, \gamma_0; 0]$ are distinct. Using Proposition 6.4, we factor $[\gamma_0, \gamma_0; 0]$ to obtain

$$[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!] = [\gamma_0, \gamma_1] \cup [\![\gamma_1, \gamma_0]\!] \cup [\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!].$$

The relative Euler classes on the right-hand side of the equation, in order, are $\pm\gamma_0 \pm \gamma_1$, $\pm(\gamma_0 + \gamma_1)$, $\gamma_1 - \gamma_0$. (The reason we have $\pm(\gamma_0 + \gamma_1)$

for the second term is due to the relative orientations of γ_0 and γ_1.) For the union of the second and third layers to be tight, the second layer must have relative Euler class $\gamma_0 + \gamma_1$ in order to cancel the γ_0's (and the first layer must be $-(\gamma_0 + \gamma_1)$). Therefore,

$$\begin{aligned}&[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!]\\ &= [\gamma_0, \gamma_1; -(\gamma_0 + \gamma_1)] \cup [\![\gamma_1, \gamma_0; \gamma_0 + \gamma_1]\!] \cup [\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!].\end{aligned}$$

Applying the same calculation to $[\![\gamma_0, \gamma_1; \gamma_0 - \gamma_1]\!]$, we see that the two non-I-invariant tight contact structures $[\gamma_0, \gamma_0; 0]$ can be distinguished by the factorization into $[\gamma_0, \gamma_1] \cup [\![\gamma_1, \gamma_0]\!]$.

It remains to dig further to obtain the remaining two tight contact structures $[\gamma_0, \gamma_0]$ with $\mathrm{PD}(\widetilde{e}) = \pm 2\gamma_0$. It suffices to factor $[\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!]$ into

$$[\gamma_0, \gamma_0; -2\gamma_0] \cup [\![\gamma_0, \gamma_1; \gamma_0 - \gamma_1]\!] \cup [\![\gamma_1, \gamma_0; \gamma_0 + \gamma_1]\!] \cup [\![\gamma_0, \gamma_1; \gamma_1 - \gamma_0]\!].$$

Similarly, we obtain $[\gamma_0, \gamma_0; 2\gamma_0]$. q.e.d.

6.3 Proof of Parts (1) and (2) of Theorem 1.1 and of Theorem 6.1

We first need the following lemma:

Lemma 6.6. *If $\gamma_0 \neq \gamma_1$, then $[\gamma_0, \gamma_1]$ contains a basic slice.*

Proof. First suppose that $|\gamma_0 \cap \gamma_1| \neq 0$. Then we can realize $\gamma_1 \times \{0, 1\}$ on $\Sigma_0 \sqcup \Sigma_1$ by efficient Legendrian curves, apply the Imbalance Principle to the convex annulus with boundary $\gamma_1 \times \{0, 1\}$, and find a bypass along $\gamma_1 \times \{0\} \subset \Sigma_0$ from the interior of $\Sigma \times I$. Let $\alpha \subset \gamma_1 \times \{0\}$ be the Legendrian arc of attachment for the bypass. There are two possibilities for α:

(i) α starts on a dividing curve γ_0^1 (one of the two curves parallel to γ_0 on Σ_0), passes through the other curve γ_0^2, and ends on γ_0^1;

(ii) α starts on γ_0^1, passes through γ_0^2, and ends on γ_0^2 after going through a nontrivial loop.

(i) is clearly an attachment which gives rise to a basic slice. For (ii), let $P \subset \Sigma_0$ be a pair-of-pants neighborhood of the union of α and the annulus bounded by γ_0^1 and γ_0^2. We write $\partial P = \gamma \sqcup B(\gamma) \sqcup \delta$, where γ is parallel to γ_0^i and $B(\gamma)$ is parallel to the dividing set of $\Sigma_{1/2}$, obtained by isotoping Σ_0 through the bypass attached along α. See the bottom illustration in Figure 20 for the pair-of-pants P.

We claim that γ and $B(\gamma)$ are not isotopic. If they were, then they would cobound an annulus B, and $B \cup P$ would be a once-punctured torus. In a once-punctured torus, an efficient, nontrivial arc or closed curve will intersect another only in positive intersections or only in negative intersections. This contradicts the efficiency of the original curve $\gamma_1 \times \{0\}$.

Therefore, we may shrink $\Sigma \times [0,1]$ and assume that $\gamma = \gamma_0 \times \{0\}$ and $B(\gamma) = \gamma_1 \times \{0\}$ are disjoint and nonisotopic. Let P be a pair-of-pants $\subset \Sigma_0$, where $\partial P = \gamma \sqcup B(\gamma) \sqcup \delta$.

Assume δ is nonseparating. If γ and δ lie on the same connected component of $\Sigma_0 \setminus \mathrm{int}(P)$, let β_1 be an arc in P connecting γ and δ and let β_2 be an arc in $\Sigma_0 \setminus \mathrm{int}(P)$ connecting the same points on γ and δ as β_1. (See Figure 21, Type C_1.) Now take $\beta = \beta_1 \cup \beta_2$. Since β does not intersect $B(\gamma)$, the Imbalance Principle applied to $\beta \times [0,1]$ produces a bypass of Type (i), and hence a basic slice. If $B(\gamma)$ and δ lie on the same connected component of $\Sigma_0 \setminus \mathrm{int}(P)$, an analogous argument produces β that intersects $B(\gamma)$ once and does not intersect γ, and another application of the Imbalance Principle produces a basic slice. (See Figure 21, Type C_2.)

If δ is separating, denote the component of $\Sigma_0 \setminus P$ it bounds by S_1. Let β_1 be a nonseparating arc in S_1 which starts and ends on δ, and let β_2 be a nonseparating arc in another component of $\Sigma_0 \setminus P$ which starts and ends on γ. Let β be a closed nonseparating curve obtained by joining those arcs by arcs in P (Figure 21, Types C_3 and C_4). Note that a nonseparating β_2 exists and β can be chosen efficient and Legendrian in both Σ_0 and Σ_1 because γ and $B(\gamma)$ are not isotopic. Applying the Imbalance Principle to this β produces a bypass of Type (ii) with nonseparating δ. This in turn, we have shown, contains a basic slice.
q.e.d.

Note. A similar but slightly more involved argument will be carried out in Proposition 7.2. The figures we refer to are the same as the ones we need for that argument.

Now that we know that $[\gamma_0, \gamma_1]$ contains a basic slice, we may apply Proposition 3.5, together with Proposition 6.2, to factor:

$$\begin{aligned}[\gamma_0, \gamma_1] &= [\![\alpha_0 = \gamma_0, \alpha_1; \alpha_1 - \alpha_0]\!] \cup [\![\alpha_1, \alpha_2]\!] \cup \dots \\ &\quad \dots \cup [\![\alpha_{k-2}, \alpha_{k-1}]\!] \cup [\alpha_{k-1}, \alpha_k = \gamma_1],\end{aligned}$$

subject to the following:

(1) α_i, $i = 1, \dots, k$, are oriented nonseparating curves.

(2) $|\alpha_i \cap \alpha_{i+1}| = 1$ $(i = 0, \dots, k-1)$ and $|\alpha_i \cap \alpha_{i+2}| = 0$ $(i = 0, \dots, k-2)$.

(3) $\langle \alpha_{i+1}, \alpha_i \rangle = 1$, $i = 0, \dots, k-1$.

(4) All the slices except for the last are basic slices.

(5) Without loss of generality, $\mathrm{PD}(\widetilde{e})$ of the first factor is $\alpha_1 - \alpha_0$.

We will inductively prove that the rest of the $\mathrm{PD}(\widetilde{e})$'s must be, in order, $\alpha_2 - \alpha_1, \alpha_3 - \alpha_2, \dots, \alpha_{k-1} - \alpha_{k-2}, \pm\alpha_k - \alpha_{k-1}$.

Lemma 6.7. *Suppose* $[\alpha_{i-1}, \alpha_{i+1}]$ *is the union* $[\![\alpha_{i-1}, \alpha_i]\!] \cup [\![\alpha_i, \alpha_{i+1}]\!]$ *of two basic slices with* $\langle \alpha_i, \alpha_{i-1} \rangle = \langle \alpha_{i+1}, \alpha_i \rangle = +1$, $|\alpha_{i-1} \cap \alpha_{i+1}| = 0$, *and* $\mathrm{PD}(\widetilde{e}([\![\alpha_{i-1}, \alpha_i]\!])) = \alpha_i - \alpha_{i-1}$. *If* $[\alpha_{i-1}, \alpha_{i+1}]$ *is tight, then* $\mathrm{PD}(\widetilde{e}([\![\alpha_i, \alpha_{i+1}]\!])) = \alpha_{i+1} - \alpha_i$.

Proof. Since $[\![\alpha_i, \alpha_{i+1}]\!]$ is a basic slice, we know that

$$\mathrm{PD}(\widetilde{e}([\![\alpha_i, \alpha_{i+1}]\!])) = \pm(\alpha_{i+1} - \alpha_i).$$

If $\mathrm{PD}(\widetilde{e}([\![\alpha_i, \alpha_{i+1}]\!])) = -\alpha_{i+1} + \alpha_i$, then

$$\mathrm{PD}(\widetilde{e}([\alpha_{i-1}, \alpha_{i+1}])) = -\alpha_{i-1} + 2\alpha_i - \alpha_{i+1}. \tag{3}$$

To obtain a contradiction, we use the fact that $|\alpha_{i-1} \cap \alpha_{i+1}| = 0$ and calculate the possible $\mathrm{PD}(\widetilde{e}([\alpha_{i-1}, \alpha_{i+1}]))$ using a different method. If γ is any closed curve with $|\gamma \cap \alpha_{i-1}| = |\gamma \cap \alpha_{i+1}| = 0$, then $\langle \widetilde{e}([\alpha_{i-1}, \alpha_{i+1}]), \gamma \times I \rangle = 0$. There are now two possibilities: either α_{i-1} and α_{i+1} are homologous or they are not. If they are homologous, then there are $2g-1$ generators γ for $H_1(\Sigma; \mathbb{Z})$ satisfying $|\gamma \cap \alpha_{i-1}| = |\gamma \cap \alpha_{i+1}| = 0$ and which therefore evaluate to zero. If γ is a closed curve satisfying $|\gamma \cap \alpha_{i-1}| = |\gamma \cap \alpha_{i+1}| = 1$, then $\langle \widetilde{e}([\alpha_{i-1}, \alpha_{i+1}]), \gamma \times I \rangle = \pm 2$ or 0, depending on the signs of the ∂-parallel components. Hence,

$$\mathrm{PD}(\widetilde{e}([\alpha_{i-1}, \alpha_{i+1}])) = \pm\alpha_{i-1} \pm \alpha_{i+1} = \pm 2\alpha_{i-1} \text{ or } 0. \tag{4}$$

Next, if α_{i-1} and α_{i+1} are not homologous, then there are $2g-2$ generators γ for $H_1(\Sigma; \mathbb{Z})$ satisfying $|\gamma \cap \alpha_{i-1}| = |\gamma \cap \alpha_{i+1}| = 0$ and which therefore evaluate to zero. There are two other basis elements

γ, γ' of $H_1(\Sigma;\mathbb{Z})$ which satisfy $|\gamma\cap\alpha_{i-1}|=0$, $|\gamma\cap\alpha_{i+1}|=1$, and $|\gamma'\cap\alpha_{i-1}|=1$, $|\gamma'\cap\alpha_{i+1}|=0$. We evaluate $\langle\widetilde{e}([\alpha_{i-1},\alpha_{i+1}]),\gamma\times I\rangle=\pm1$ and $\langle\widetilde{e}([\alpha_{i-1},\alpha_{i+1}]),\gamma'\times I\rangle=\pm1$, which give

$$\mathrm{PD}(\widetilde{e}([\alpha_{i-1},\alpha_{i+1}]))=\pm\alpha_{i-1}\pm\alpha_{i+1}. \tag{5}$$

It now suffices to note that Equation (3) is in contradiction with Equations (4) or (5) — simply intersect with α_{i-1}. Therefore, we are left with $\mathrm{PD}(\widetilde{e}([\![\alpha_i,\alpha_{i+1}]\!]))=\alpha_{i+1}-\alpha_i$. q.e.d.

Thus, by Lemma 6.7, we find that the $\mathrm{PD}(\widetilde{e})$'s of the basic slices are $\alpha_2-\alpha_1,\alpha_3-\alpha_2,\dots,\alpha_{k-1}-\alpha_{k-2}$. Finally, although the last slice is not a basic slice, an argument almost identical to that of Lemma 6.7 proves that the relative Euler class is $\pm\alpha_k-\alpha_{k-1}$. Therefore, we see that the initial basic slice $[\![\alpha_0,\alpha_1]\!]$ uniquely determines all the subsequent basic slices and reduces the possibilities for the last slice to two. Thus, there are at most 4 possibilities for $[\gamma_0,\gamma_1]$, up to isotopy rel boundary. Adding up the relative Euler classes of the slices, we obtain $\mathrm{PD}(\widetilde{e}([\gamma_0,\gamma_1]))=\pm\gamma_0\pm\gamma_1$. The relative Euler classes distinguish the 4 possibilities, provided γ_0 and γ_1 are not homologous.

We now have the following proposition:

Proposition 6.8. *Suppose $[\alpha,\alpha']$ is a tight contact structure, where α and α' are nonseparating. Then $\mathrm{PD}(\widetilde{e}([\alpha,\alpha']))=\pm\alpha\pm\alpha'$. Moreover, if $[\alpha,\alpha']$ admits a factorization $[\![\alpha_0=\alpha,\alpha_1]\!]\cup[\![\alpha_1,\alpha_2]\!]\cup\cdots\cup[\![\alpha_{k-1},\alpha_k=\alpha']\!]$ with $\langle\alpha_{i+1},\alpha_i\rangle=+1$, then $\mathrm{PD}(\widetilde{e}([\alpha,\alpha']))=\pm(\alpha'-\alpha)$.*

Proof. This was largely proved in the above paragraphs, with the difference that we required that $|\alpha_i\cap\alpha_{i+2}|=0$. This extra condition is not required in Proposition 6.8, since there always exists a subdivision which satisfies this extra property. q.e.d.

Theorem 6.1 immediately follows from Proposition 6.8. The following two lemmas complete the proof of Theorem 1.1.

Lemma 6.9. *All four $[\gamma_0,\gamma_1]$ are $\Sigma\times I$ layers inside some basic slice $[\![\gamma_0,\gamma]\!]$ with $\langle\gamma,\gamma_0\rangle=+1$.*

Proof. We will start with $[\![\gamma_0,\gamma;\gamma-\gamma_0]\!]$ and find two of the four possibilities; the other two can be found inside $[\![\gamma_0,\gamma;-\gamma+\gamma_0]\!]$. Using Proposition 6.2, we find a factorization:

$$[\![\gamma_0,\gamma;\gamma-\gamma_0]\!]=[\![\alpha_0=\gamma_0,\alpha_1]\!]\cup[\![\alpha_1,\alpha_2]\!]\cup\cdots\cup[\![\alpha_{k-1},\alpha_k=\gamma_1]\!]\cup[\alpha_k,\gamma].$$

The tight contact structure $[\gamma_0, \gamma_1] \subset [\gamma_0, \gamma]$, obtained by layering

$$[\![\alpha_0, \alpha_1]\!], \dots, [\![\alpha_{k-1}, \alpha_k]\!],$$

must have $\mathrm{PD}(\widetilde{e}) = \gamma_1 - \gamma_0$ by Proposition 6.8. Now, by Theorem 6.1, if $\mathrm{PD}(\widetilde{e}([\gamma_0, \gamma])) = \gamma - \gamma_0$, then $\mathrm{PD}(\widetilde{e}([\gamma_0, \gamma_1]))$ must be $\gamma_1 - \gamma_0$. To obtain $-(\gamma_0 + \gamma_1)$, we start with $[\![\gamma_0, \gamma; \gamma - \gamma_0]\!]$ and factor:

$$\begin{aligned} [\![\gamma_0, \gamma; \gamma - \gamma_0]\!] = [\![\alpha_0 = \gamma_0, \alpha_1]\!] \cup [\![\alpha_1, \alpha_2]\!] \cup \dots \\ \cup [\![\alpha_{k-1}, \alpha_k = \gamma_1]\!] \cup [\![\alpha_k, \gamma]\!] \cup [\![\gamma, \alpha_k]\!] \cup [\alpha_k, \gamma], \end{aligned}$$

with all but the last layer basic. Throwing away the last layer on the right-hand side, we obtain $[\gamma_0, \gamma_1]$ with $\mathrm{PD}(\widetilde{e}) = -(\gamma_0 + \gamma_1)$. q.e.d.

Lemma 6.10 (Unique factorization). *The 4 tight contact structures on $[\gamma_0, \gamma_1]$ are distinct.*

Proof. If γ_0 and γ_1 are not homologous, then the four tight contact structures are distinguished by the relative Euler class. If $\gamma_0 \neq \gamma_1$ are homologous, then the relative Euler class cannot distinguish between $\gamma_1 - \gamma_0$ and $\gamma_0 - \gamma_1$. We already showed that one of the $[\gamma_0, \gamma_1]$ admits a factorization:

$$\begin{aligned} [\gamma_0, \gamma_1; \gamma_1 - \gamma_0] = [\![\alpha_0 = \gamma_0, \alpha_1; \alpha_1 - \alpha_0]\!] \cup [\![\alpha_1, \alpha_2; \alpha_2 - \alpha_1]\!] \cup \dots \\ \cup [\![\alpha_{k-1}, \alpha_k = \gamma_1; \alpha_k - \alpha_{k-1}]\!], \end{aligned}$$

with $\langle \alpha_{i+1}, \alpha_i \rangle = +1$. We claim that given any other factorization:

$$[\gamma_0, \gamma_1; \gamma_1 - \gamma_0] = [\![\alpha_0 = \gamma_0, \alpha_1]\!] \cup [\![\alpha_1, \alpha_2]\!] \cup \dots \cup [\![\alpha_{k-1}, \alpha_k = \gamma_1]\!],$$

each $[\![\alpha_i, \alpha_{i+1}]\!]$ has relative Euler class $\alpha_{i+1} - \alpha_i$. From the discussion in Lemma 6.7, we see that it suffices to prove that the relative Euler class of $[\![\alpha_{k-1}, \alpha_k]\!]$ is $\alpha_k - \alpha_{k-1}$. But now, by Lemma 6.9, we see that if α_{k+1} satisfies $\langle \alpha_{k+1}, \alpha_k \rangle = 1$, then $[\gamma_0, \gamma_1] \cup [\![\alpha_k, \alpha_{k+1}; \alpha_{k+1} - \alpha_k]\!]$ is tight. Now, applying Lemma 6.7, we see that $[\![\alpha_{k-1}, \alpha_k]\!]$ must have relative Euler class $\alpha_k - \alpha_{k-1}$. q.e.d.

7. Classification of tight contact structures on hyperbolic 3-manifolds which fiber over the circle

In this section we provide the proof of Theorem 1.2. Let M be a closed, oriented 3-manifold which fibers over the circle with fibers (oriented surfaces Σ) of genus $g \geq 2$ and pseudo-Anosov monodromy $f : \Sigma \to \Sigma$. In other words, $M = \Sigma \times [0,1]/\sim$, where $(x,0) \sim (f(x),1)$. Recall that the pseudo-Anosov condition is equivalent to saying that for every multicurve $\Gamma \subset \Sigma$, $f(\Gamma) \neq \Gamma$. The assumption that $e(\xi)$ is extremal guarantees that the dividing set on any convex fiber Σ is a union of pairs of parallel curves bounding annuli. To prove Theorem 1.2, we first show that there exists a convex fiber Σ for which Γ_Σ consists of exactly two nonseparating curves. This is accomplished by starting with an arbitrary convex fiber Σ and inductively reducing the number of curves in Γ_Σ by two, by isotoping Σ through an appropriate bypass. The following proposition will be used to show the existence of an appropriate bypass.

Proposition 7.1. *Let ξ be a tight contact structure on $\Sigma \times [0,1]$ with convex boundary, and suppose $\Gamma_{\Sigma_0} \neq \Gamma_{\Sigma_1}$. Then there is a closed efficient curve γ, possibly separating, such that $|\gamma \cap \Gamma_{\Sigma_0}| \neq |\gamma \cap \Gamma_{\Sigma_1}|$.*

Proof. Suppose the dividing set Γ_{Σ_0} is the disjoint union, over $i = 1, \dots, k$, of $m_i > 0$ curves isotopic to δ_i, and Γ_{Σ_1} is the disjoint union, over $j = 1, \dots, l$, of $n_j > 0$ curves isotopic to ε_j. After an isotopy, we may assume that all δ_i and ε_j intersect transversely and efficiently (realize the geometric intersection number). If $\delta_i \cap \varepsilon_j \neq \emptyset$ for some i and j, then letting $\gamma = \delta_i$ satisfies the conclusion of the proposition.

We may now assume that $\delta_i \cap \varepsilon_j = \emptyset$ for all pairs (i,j). Thus $\{\delta_i\}_{i=1}^k \cup \{\varepsilon_j\}_{j=1}^l$ may be completed to a pair-of-pants decomposition of Σ, after throwing away duplicates, if δ_i and ε_j are isotopic. It is not hard to show directly that weights on the cuffs of a pair-of-pants decomposition are determined by their intersection number with transverse embedded curves; thus the required curve γ exists. (For more details, see [15].) q.e.d.

The following grew out of discussions with John Etnyre:

Proposition 7.2. *Let ξ be a tight contact structure on $M = \Sigma \times I/\sim$ with $\langle e(\xi), \Sigma\rangle = \pm(2g-2)$. Then there exists a convex surface isotopic to the fiber whose dividing set consists of two parallel nonseparating curves.*

Proof. Cut M along a convex surface isotopic to the fiber, and denote the cut-open manifold $\Sigma \times [0,1]$ and the contact structure $\xi|_{\Sigma\times[0,1]}$ by ξ. Then $\Gamma_{\Sigma_1} = f(\Gamma_{\Sigma_0})$. We assume that $\#\Gamma_{\Sigma_0} = \#\Gamma_{\Sigma_1} > 2$, since otherwise we are done.

By Proposition 7.1, we may choose γ to be an efficient curve such that there is an imbalance in the number of intersections of $\partial(\gamma \times I)$ with Γ_{Σ_0} and Γ_{Σ_1}. Note that if γ is separating, then we may need to use Lemma 2.4 (or its proof) to Legendrian realize $\partial(\gamma \times I)$. There is a bypass contained in $\Sigma \times I$, attached along either Σ_0 or Σ_1. We can assume without loss of generality that the bypass is along Σ_0. Since γ is chosen to be efficient, the bypass can neither be trivial nor increase the number of dividing curves on Σ_0.

Denote the attaching arc of the bypass by α. We will discuss the possible types of bypass attachments, and in each case show that the number of dividing curves can eventually be decreased by two.

Type A. The attaching arc α intersects three distinct dividing curves.

In this case, there exists a consecutive parallel pair of dividing curves which intersect α. By isotoping Γ_{Σ_0} through the bypass, we will therefore reduce the number of dividing curves by two. See Figure 18.

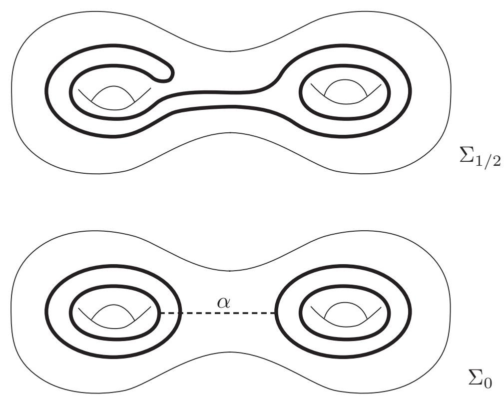

Figure 18: Type A attaching arc.

Type B. The attaching arc α starts on a dividing curve γ_1, passes through a parallel dividing curve γ_2, and ends on γ_1.

Isotop Σ_0 through this bypass to obtain $\Sigma_{1/2}$. Let $N \subset \Sigma$ be a punctured torus so that $N \times \{0\}$ is a regular neighborhood of the union of α and the annulus bounded by γ_1 and γ_2 (and has convex boundary). Identify the region between Σ_0 and $\Sigma_{1/2}$ with $\Sigma \times [0, \frac{1}{2}]$ in such a way that the contact structure is I-invariant on $(\Sigma \setminus N) \times [0, \frac{1}{2}]$ and is given by a single bypass attachment on $N \times [0, \frac{1}{2}]$. See Figure 19(A).

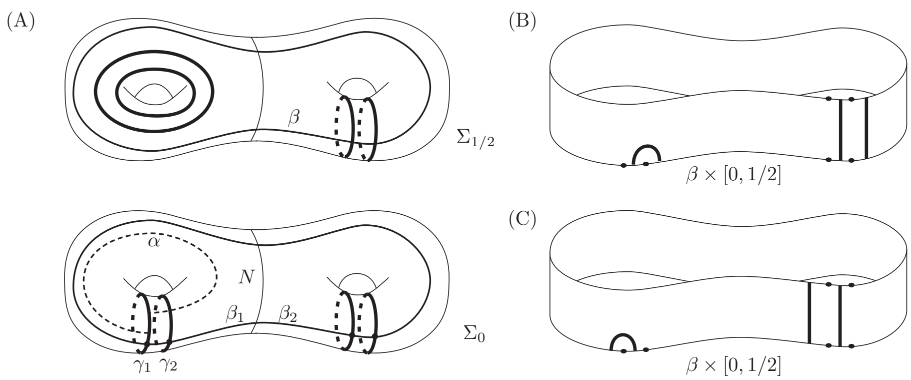

Figure 19: Type B attaching arc.

Since $\#\Gamma_{\Sigma_0} = \#\Gamma_{\Sigma_{1/2}} > 2$, there are dividing curves contained in $\Sigma \setminus N$. Choose $\beta \subset \Sigma$ to be a closed curve formed out of an arc $\beta_1 \subset N$ and an arc $\beta_2 \subset \Sigma \setminus N$, where the arcs have the following properties:

(i) $\beta_1 \times \{0\}$ intersects each of γ_1 and γ_2 once,

(ii) $\beta_1 \times \{\frac{1}{2}\}$ intersects no dividing curves, and

(iii) β_2 has nontrivial, efficient intersections with dividing curves in $\Sigma \setminus N$.

We may choose $\beta \times [0, \frac{1}{2}]$ to be convex. Since the contact structure is I-invariant on $(\Sigma \setminus N) \times [0, \frac{1}{2}]$, the dividing curves along β_2 are vertical. Thus there are only two possible dividing curve configurations, both of which are shown in Figure 19(B,C), and either of these forces the existence of a bypass of Type A along a subarc of $\beta \times \{0\}$.

Type C. The attaching arc α starts on a dividing curve γ_1, passes through a parallel dividing curve γ_2, and ends on γ_2 after going around a nontrivial loop.

Let $P \subset \Sigma$ be a pair-of-pants so that $P_0 = P \times \{0\}$ is a regular neighborhood of the union of α and the annulus bounded by γ_1 and γ_2. We write $\partial P = \gamma \sqcup B(\gamma) \sqcup \delta$, where γ is parallel to γ_1 and γ_2 and $B(\gamma)$ is parallel to the two new dividing curves obtained from γ_1 and γ_2 after isotoping through the bypass along α. See Figure 20. We first observe that γ, $B(\gamma)$ and δ are all essential curves on Σ. γ was assumed to be essential, and the fact that the attaching arc α was nontrivial and non-$\#\Gamma$-increasing implies that $B(\gamma)$ and δ are essential. Also, γ and $B(\gamma)$ are not isotopic, since otherwise they would cobound an annulus and $B \cup P$ would be a once-punctured torus, contradicting the efficiency of the attaching arc.

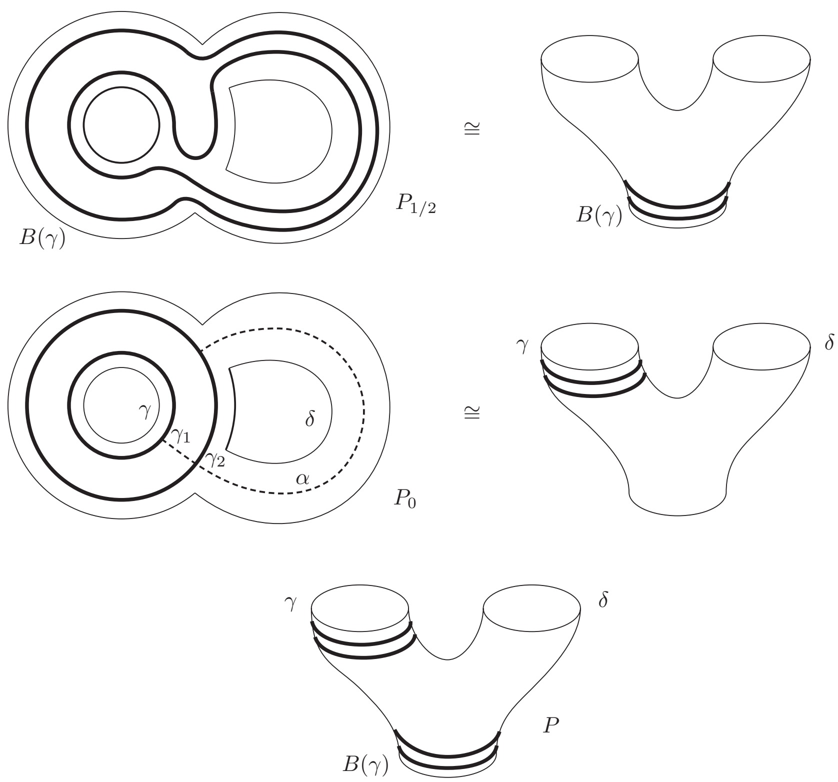

Figure 20: Type C attaching arc.

Let $\Sigma_{1/2}$ be the convex surface obtained by isotoping Σ_0 through the bypass along α. Identify the region between Σ_0 and $\Sigma_{1/2}$ with $\Sigma \times [0, \frac{1}{2}]$ in such a way that the contact structure is I-invariant on $(\Sigma \setminus P) \times [0, \frac{1}{2}]$.

The proof now proceeds roughly as in the Type B case, that is, we produce a curve β so that the Imbalance Principle can be applied to $\beta \times [0, \frac{1}{2}]$. To do this, we must consider the possible ways that P can sit in Σ. See Figure 21.

Type $\mathbf{C_1}$. δ is nonseparating and there is an arc $\beta_2 \subset \Sigma \setminus P$ connecting γ and δ.

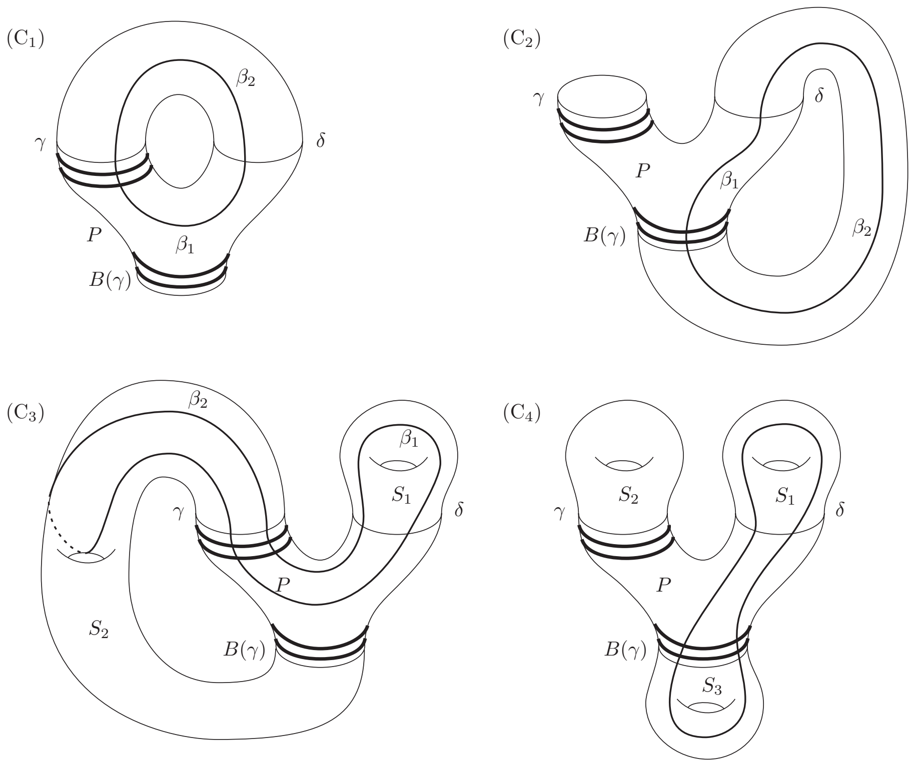

Figure 21: Type C cases.

Let $\beta_1 \subset P$ be an arc connecting the same points of γ and δ as β_2, and let $\beta = \beta_1 \cup \beta_2$. The arcs β_1, β_2 may be chosen so that $\beta \times \{0\}$ and $\beta \times \{\frac{1}{2}\}$ are efficient with respect to Γ_0 and $\Gamma_{1/2}$. If β_2 intersects any dividing curves of $\Sigma \setminus P$, then the Imbalance Principle applied to $\beta \times [0, \frac{1}{2}]$ produces a bypass of Type A. Otherwise, since β is nonseparating, $\beta \times \{\frac{1}{2}\}$ is nonisolating and can be made a Legendrian divide. Again the Imbalance Principle applied to $\beta \times [0, \frac{1}{2}]$ produces a bypass, this time a degenerate one which can be perturbed into a bypass of Type B.

Type $\mathbf{C_2}$. δ is nonseparating and there is an arc $\beta_2 \subset \Sigma \setminus P$ connecting $B(\gamma)$ and δ.

Let $\beta_1 \subset P$ be an arc connecting the same points of $B(\gamma)$ and δ as β_2 and let $\beta = \beta_1 \cup \beta_2$. As in Type C_1, the Imbalance Principle applied to β produces a bypass of Type A or B.

Type C_3. δ is separating and there is an arc contained in $\Sigma \setminus P$ connecting $B(\gamma)$ and γ.

Let S_1 be the component of $\Sigma \setminus P$ where $\partial S_1 = \delta$, and let S_2 be the component where $\partial S_2 = \gamma \sqcup B(\gamma)$. S_1 must have negative Euler characteristic because δ cannot bound a disk, and S_2 must have negative Euler characteristic since γ and $B(\gamma)$ are not isotopic.

Let $\beta_1 \subset S_1$ be a nonseparating arc starting and ending on δ, and let $\beta_2 \subset S_2$ be a nonseparating arc starting and ending on γ. Since $\#\Gamma_{\Sigma_0} = \#\Gamma_{\Sigma_{1/2}} > 2$, there must be dividing curves contained in $\Sigma \setminus P = S_1 \sqcup S_2$. We may choose the arcs β_1, β_2 so that at least one of them has nontrivial, efficient intersection with $\Gamma_{\Sigma\setminus P}$. Pick two disjoint arcs in P connecting the endpoints of β_1 to the endpoints of β_2 and let β be the union of all four arcs. The Imbalance Principle produces either a bypass of Type A or, since the β_i were chosen to be nonseparating, a bypass of Type C_1 or C_2.

Type C_4. All three curves δ, γ, and $B(\gamma)$ are separating.

Let S_1, S_2 and S_3 be, in order, the components of $\Sigma \setminus P$ bounded by δ, γ, and $B(\gamma)$. Since each S_i must have genus greater than 0, the proof is the same as in Type C_3, with one possible extra case. If all of the dividing curves of $\Sigma \setminus P$ are in S_3, then, to force $\beta \times \{\frac{1}{2}\}$ to be isolating, the desired β must be produced in $S_3 \cup P \cup S_1$, and the resulting bypass will lead to a reduction in the number of dividing curves on $\Sigma_{1/2}$ instead of Σ_0.

Therefore, in all possible cases, we have found a sequence of bypasses which eventually reduces Γ_{Σ_0} to two parallel curves. We now claim that there is a convex fiber with two parallel *nonseparating* curves. Suppose Γ_{Σ_i}, $i = 0, 1$, are separating. They are not isotopic by the pseudo-Anosov property. First suppose that Γ_{Σ_0} and Γ_{Σ_1} are disjoint. Then it is easy to find an efficient nonseparating curve β which intersects Γ_{Σ_0} in four points and does not intersect Γ_{Σ_1}. Moreover, we require each arc of $\beta \setminus \Gamma_{\Sigma_0}$ to be nonseparating in $\Sigma \setminus \Gamma_{\Sigma_0}$. Then we can use the Imbalance Principle to find a bypass that transforms Γ_{Σ_0} into a pair of nonseparating curves. If Γ_{Σ_0} and Γ_{Σ_1} are not disjoint, then take β to be an efficient closed curve isotopic to a component of Γ_{Σ_1}. Applying the Imbalance Principle, we obtain $\Gamma_{\Sigma_{1/2}}$ which consists of a pair of parallel curves disjoint from and not isotopic to Γ_{Σ_0}. If $\Gamma_{\Sigma_{1/2}}$ consists of nonseparating curves, we are done; otherwise we apply the previous considerations to Γ_{Σ_0} and $\Gamma_{\Sigma_{1/2}}$. q.e.d.

Proof of Theorem 1.2. Fix an oriented nonseparating curve γ on Σ. Since f is pseudo-Anosov, γ and $f(\gamma)$ are not isotopic. We choose the orientation on $f(\gamma)$ to be the one induced from γ and f. Let us assume for simplicity that γ and $f(\gamma)$ are not homologous. (The general case, although identical in proof, is harder to state and will be left to the reader.)

Let ξ be the universally tight contact structure on $\Sigma \times [0, 1]$ with $\mathrm{PD}(\widetilde{e}(\xi)) = f(\gamma) - \gamma$. We claim that the contact structure obtained by gluing $\Sigma \times \{0\}$ with $\Sigma \times \{1\}$ via f is universally tight. It is immediate, by the Gluing Theorem 6.1, that $2n$ copies of $\Sigma \times [0, 1]$ stacked and glued together via f will produce a universally tight contact structure on $\Sigma \times [-n, n]$. It follows that the glued-up contact structure on M is universally tight, for any potential overtwisted disk in M would lift to the $\Sigma \times \mathbb{R}$ cover of M and therefore would be contained in some $\Sigma \times [-n, n]$.

To prove uniqueness, use Proposition 7.2 to choose a convex fiber Σ' such that $\Gamma_{\Sigma'} = 2\gamma'$, where γ' is some nonseparating curve, and split M along Σ' to obtain $\Sigma' \times I$. Since f is pseudo-Anosov, $\Gamma_{\Sigma' \times \{0\}} \neq \Gamma_{\Sigma' \times \{1\}}$, and therefore $\Sigma' \times I$ contains a basic slice by Lemma 6.6. It follows from Proposition 6.2 that a new convex fiber Σ'' may be chosen such that $\Gamma_{\Sigma''} = 2\gamma$.

By Theorem 1.1, there are four possibilities for $\xi_{\Sigma'' \times I}$ with distinct relative Euler classes $\pm f(\gamma) \pm \gamma$. If the relative Euler class is $\pm f(\gamma) + \gamma$, then $\Sigma'' \times [0, 1]$ contains a slice $\Sigma'' \times [0, \frac{1}{2}]$ where $\Gamma_{\Sigma'' \times \{\frac{1}{2}\}} = 2\gamma$ and the slice has relative Euler class 2γ. By peeling this slice and reattaching using f, we obtain $\Sigma'' \times [\frac{1}{2}, 1] \cup f(\Sigma'' \times [0, \frac{1}{2}])$. It follows that for any given tight contact structure on M, there always exists a convex fiber Σ such that the restriction of ξ to $\Sigma \times I$ (obtained by cutting along Σ) has relative Euler class $\pm f(\gamma) - \gamma$.

Suppose that the relative Euler class is $-f(\gamma) - \gamma$. Then choose δ to be a nonseparating curve and use the freedom of choice to split $\Sigma \times I$ along a surface Σ''' with $\Gamma_{\Sigma'''} = 2\delta$. This splitting, with an appropriate choice of orientation on δ, gives, by tightness and Theorem 6.1,

$$[\gamma, f(\gamma); -f(\gamma) - \gamma] = [\gamma, \delta; \delta - \gamma] \cup [\delta, f(\gamma); -f(\gamma) - \delta].$$

Splitting along Σ''' and reattaching using f, we see that ξ restricted to $M\backslash\Sigma'''$ is

$$[\delta, f(\gamma); -f(\gamma)-\delta] \cup f([\gamma,\delta;\delta-\gamma]) = \\ [\delta, f(\gamma); -f(\gamma)-\delta] \cup [f(\gamma), f(\delta); f(\delta)-f(\gamma)].$$

This contact structure cannot be tight by Theorem 6.1.

Thus ξ on M can always be split to have relative Euler number $f(\gamma)-\gamma$ and the uniqueness on M follows from Theorem 1.1.

It remains to discuss weak fillability. First note that there exists a C^0-small perturbation of the fibration into a universally tight contact structure ξ, by a result of Eliashberg and Thurston [7]. Since the Euler class evaluated on the fiber is unchanged under the perturbation, we have, say, $\langle e(\xi), \Sigma\rangle = -(2g-2)$. By the uniqueness which we just proved, the unique extremal tight contact structure on M with $\langle e, \Sigma\rangle = -(2g-2)$ is isotopic to ξ. Now, to prove weak symplectic fillability, we construct a symplectic 4-manifold (X,ω) with $\partial X = M$ for which $\omega|_{T\Sigma} > 0$ (for all fibers Σ). Since ξ is close to the fibration, we would then be done. The construction of X is relatively straightforward from the Lefschetz fibration perspective, once we observe that any element $f:\Sigma\to\Sigma$ of the mapping class group of a closed surface Σ can be written as a product of positive Dehn twists. (For more details on symplectic Lefschetz fibrations, see, for example [13].) We take a symplectic Lefschetz fibration $X\to D^2$ with generic fiber Σ and a singular fiber for each positive Dehn twist in the product expression. We then see that $\partial X = M$ has the desired monodromy and each fiber Σ is a symplectic submanifold. q.e.d.

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University of Southern California
Los Angeles, CA 90089

University of Georgia
Athens, GA 30602

University of Georgia
Athens, GA 30602